Matrices and their applications

Core topic:
Matrices and applications

In this chapter

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Inverse matrices and systems of linear equations

The ideas you were introduced to in chapter 3 on matrices provide a foundation for using matrices to solve simultaneous equations. The following section presents a simplified account of complex aspects of economies and industries and will serve to introduce the idea of matrix applications in a wide field of studies.

Input–output analysis

To enable government bodies and companies to plan for future needs, input–output analysis models are employed. In these models the interaction between major components of an economy is analysed so that the effect of an increase in one component can be measured against the demand for one product in other industries.

Input–output models were first developed by Nobel Prize winner, Wassily Leontief; they are used to describe economies where demand equals supply, or consumption equals production. These models can be applied to whole economies or to segments within economies.

The model, as presented here, is based on the idea that there is a finite number of goods that are produced, consumed or used as input for the same finite number of industries. Each industry produces only one type of product and can use some of its own product. Each industry and its products are interdependent. The total demand of product is the sum of the demands throughout the entire production process as well as the demand for the product by consumers.

Consider an economy comprising only 2 industries — coal and steel. One tonne of steel requires an input of 0.5 tonne of coal as well as 0.4 tonne of steel (perhaps in the form of machinery and plant). To produce 1 tonne of coal, 0.3 tonne of steel and 0.4 tonne of coal are required (perhaps to produce the steel needed for machinery). Suppose also that the total demand for steel is 18 tonnes and for coal 15 tonnes.

Let $x_1$ and $x_2$ denote the total demands for coal and steel respectively. This information can be presented on an input–output table.

<table>
<thead>
<tr>
<th>Producer</th>
<th>User</th>
<th>Final demand</th>
<th>Total demand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coal</td>
<td></td>
<td>$x_1$</td>
</tr>
<tr>
<td>Coal</td>
<td>0.4</td>
<td>0.3</td>
<td>15</td>
</tr>
<tr>
<td>Steel</td>
<td>0.5</td>
<td>0.4</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$x_2$</td>
</tr>
</tbody>
</table>
This information can be represented by a system of simultaneous linear equations:

\[
\begin{align*}
x_1 &= 0.4x_1 + 0.3x_2 + 15 \\
x_2 &= 0.5x_1 + 0.4x_2 + 18
\end{align*}
\]

and put into matrix form:

\[
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} = 
\begin{bmatrix}
0.4 & 0.3 \\
0.5 & 0.4
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix}
+ 
\begin{bmatrix}
15 \\ 18
\end{bmatrix}
\]

which can be represented as

\[
X = AX + D
\]

where \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), \( A = \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix} \) and \( D = \begin{bmatrix} 15 \\ 18 \end{bmatrix} \).

Rearrangement of the matrices (as shown below) isolates \( X \) so it can be solved.

\[
X - AX = D \\
(I - A)X = D
\]

where \( X \) post-multiplies other factors and \( I \) is the identity matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

\[
(I - A)^{-1}(I - A)X = (I - A)^{-1}D \\
X = (I - A)^{-1}D
\]

We call \( X \) the output matrix as it holds variables \( x_1 \) and \( x_2 \) that will state the total demand for steel and coal. We call \( A \) the technology matrix and \( D \) the final demand matrix. Because of its significance \( I - A \) is called the Leontief matrix.

To calculate \( X \) we first need to find \( I - A \).

\[
I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{bmatrix}
= \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}
\]

Therefore

\[
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} = \begin{bmatrix} 0.6 & -0.3 \\ -0.5 & 0.6 \end{bmatrix}^{-1}
\begin{bmatrix}
15 \\ 18
\end{bmatrix}
= \frac{1}{0.36 - 0.15}
\begin{bmatrix}
0.6 & 0.3 \\ 0.5 & 0.6
\end{bmatrix}
\begin{bmatrix}
15 \\ 18
\end{bmatrix}
= \frac{1}{0.21}
\begin{bmatrix}
14.4 \\ 18.3
\end{bmatrix}
= \begin{bmatrix} 68.57 \\ 87.14 \end{bmatrix}
\]

Thus, in order to provide the final demand of 18 tonnes of steel and 15 tonnes of coal in this economy, the steel industry must produce 87.14 tonnes of steel and the coal industry must produce 68.57 tonnes of coal.

These values can also be regarded as equilibrium values — if these values are met then the input will match the output — thus eliminating both over- and under-production.
1. Matrices can be used for input–output analysis.
2. An input–output analysis matrix can be written as
   \[ X = AX + D \]
   where
   - matrix \( X \) contains the variables to be determined and is called the output matrix
   - matrix \( A \) contains information about the input details and is called the technology matrix
   - matrix \( D \) contains information about the final demand and is called the final demand matrix.
3. The matrix equation \( X = AX + D \) can be rearranged to make \( X \) the subject:
   \[ X = (I - A)^{-1}D \]

**Inverse matrices and systems of linear equations**

1. Find the equilibrium values of \( P \) and \( Q \) in
   \[ P = Q + Y \]
   \[ Q = x + yP \]
   where \( Y = $6 \) million, \( x = $15 \) million and \( y = 0.5 \).

2. Find the equilibrium values of
   \[ x = 45 - 2y \]
   \[ x = 0.6y \]

3. The equilibrium state for 2 commodities is given by
   \[ 2a - 3b = 35 \]
   \[ -3a + 2b = 40 \]
   Find the equilibrium values for these goods.

4. Consider the simplified Keynesian system
   \[ Y = C + 30 \]
   \[ C = 50 + 0.8Y \]
   where \( Y \) denotes national income and \( C \) denotes consumption. Find the equilibrium values for \( Y \) and \( C \).

5. Find the equilibrium values for the supply and demand model
   \[ x = 21 - 4y \quad \text{(demand)} \]
   \[ x = 15 - 3y \quad \text{(supply)} \]

6. The technology matrix for an economy which produces only 2 commodities is given by
   \[ A = \begin{bmatrix} 0.2 & 0.6 \\ 0.1 & 0.5 \end{bmatrix} \]
   and the final demand matrix \( D = \begin{bmatrix} 25 \\ 36 \end{bmatrix} \).
   Solve \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \).
7 A simple economy is shown in the table.

<table>
<thead>
<tr>
<th>Producer</th>
<th>User</th>
<th>Final demand</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>p</td>
<td>0.25</td>
<td>0.4</td>
<td>30</td>
</tr>
<tr>
<td>q</td>
<td>0.3</td>
<td>0.5</td>
<td>25</td>
</tr>
</tbody>
</table>

Find the total demand matrix.

8 A certain economy consists of 2 industries, mining ore and manufacturing. The production of 1K dollars (K = 1000) of manufactured goods requires 0.6K dollars of manufactured goods and 0.15K dollars of ore, while the production of 1K dollars of ore requires 0.4K dollars of manufactured goods and 0.3K dollars of ore.

The final demand for manufacturing and mining ore is 120K dollars and 145K dollars respectively.

a Prepare a matrix table for this information.
b Represent the information in matrix form, using $X$ as the output matrix, $A$ the technology matrix and $D$ the demand matrix.
c Find $(I - A)^{-1}$.
d Find the total demand matrix (output matrix) $X$.
e Verify that $D = (I - A)X$.

---

**Solving simultaneous equations — the Gaussian elimination method**

The example based on the economy with only 2 industries used in the previous section is obviously a little unrealistic. However, if this economy were to involve more than 2 industries we have no tools to solve this type of problem, at this stage. This is because it would involve finding the inverse of a $3 \times 3$ matrix using the example of an economy with 3 industries, whereas we are limited to finding the inverse of a $2 \times 2$ matrix.

A variety of methods can be used to solve a system of simultaneous equations, such as those generated in the previous section. We will concentrate at this stage on the method known as *Gaussian elimination*. In future sections of this chapter we will introduce 2 more methods, as well as graphics calculator techniques for solving simultaneous equations.

The Gaussian elimination method is described below.

Consider a set of 2 simultaneous linear equations in 2 unknowns, $x$ and $y$ as given:

$$ax + by = u$$
$$cx + dy = v$$

As we saw earlier, this system can be represented as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$B = \begin{bmatrix} u \\ v \end{bmatrix}$$
$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A$ is the *coefficient matrix*, $B$ is the *constant matrix* and $X$ is the *variable or unknown matrix*.
We can combine matrices $A$ and $B$ as \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
\] where this is referred to as an augmented matrix, denoted by $[A \mid B]$.

The following example shows the familiar algebraic process involved in solving simultaneous equations, only this time it is in matrix form. These steps will work towards producing the identity matrix $I$ on the left-hand side and then the solution will be on the right-hand side.

$[A \mid B] \rightarrow [I \mid X]$

**WORKED Example 1**

Solve this system of simultaneous linear equations:

\[
\begin{align*}
x + 2y &= 3 \\
2x + 3y &= 5
\end{align*}
\]

**THINK**

1. Set equations in matrix form.

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}
\]

2. Convert to augmented matrix form.

$[A \mid B] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

3. The first step will be to eliminate the $a_{21}$ element to produce a 0.

The ‘$R_2 - 2R_1$’ dialogue denotes row 2 minus twice row 1 (applied to row 2).

\[
R_2 - 2R_1: \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}
\]

4. Next we need to make $a_{12}(2) = 0$. We can do this by adding row 1 and twice row 2 (applied to row 1).

\[
R_1 + 2R_2: \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}
\]

5. Finally, multiply row 2 by $-1$ so that $a_{22} = 1$.

\[
R_2 \times -1: \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

6. $[A \mid B]$ now resembles $[I \mid X]$ and $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

\[
[A \mid B] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

\[
X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ that is } x = 1 \text{ and } y = 1
\]

7. Verify your answers by substitution into the original equations.

\[
1 + 2(1) = 3 \quad \text{(true)} \\
2(1) + 3(1) = 5 \quad \text{(true)}
\]

All the communication given to the left of each matrix should be included in your own work as justification for how you have proceeded from one step to the other, and also for your own reference.
The following procedures are used in this method:
1. multiply a row by a constant
2. add and subtract one row to or from another
3. work down to position zeros where you want them in \( I \)
4. ‘roll back’ lower rows up the matrix to position zeros above the leading diagonal
5. when trying to introduce 1s in the heading diagonal, look at the rows below; if a 1 already exists, the rows can be interchanged.

*Note:* Always put zeros where you need them *under the leading diagonal* before you start putting zeros above the leading diagonal.

Let’s expand this method to work with a \( 3 \times 3 \) matrix.

**WORKED Example 2**

Solve this system of linear equations.

\[
\begin{align*}
  x + 2y + z &= 8 \\
  x - y + z &= 9 \\
  x + y + 2z &= 4
\end{align*}
\]

**THINK**

1. Set the equations in matrix form.

\[
A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 9 \\ 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

2. Convert to augmented matrix form.

\[
[A | B] = \begin{bmatrix} 1 & 2 & 1 & 8 \\ 1 & -1 & 1 & 9 \\ 1 & 1 & 2 & 4 \end{bmatrix}
\]

3. The first element to eliminate is \( a_{31} = 1 \) so use \( R_3 - R_1 \) (in row 3).

\[
R_3 - R_1: \quad \begin{bmatrix} 1 & 2 & 1 & 8 \\ 1 & -1 & 1 & 9 \\ 0 & -1 & 1 & -4 \end{bmatrix}
\]

4. Next work on \( a_{21} = 1 \), directly above \( a_{31} \) so \( R_2 - R_1 \) (in row 2).

\[
R_2 - R_1: \quad \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & 1 \\ 0 & -1 & 1 & -4 \end{bmatrix}
\]

5. Next eliminate \( a_{32} = 1 \) so use \( R_3 - \frac{1}{3} R_2 \) (in row 3).

\[
R_3 - \frac{1}{3} R_2: \quad \begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix}
\]

6. Next roll row 3 back up to eliminate \( a_{13} \) (in row 1).

\[
R_1 - R_3: \quad \begin{bmatrix} 1 & 2 & 0 & 12 \frac{1}{3} \\ 0 & -3 & 0 & -1 \\ 0 & 0 & 1 & -4 \frac{1}{3} \end{bmatrix}
\]

Continued over page
This method can be quite frustrating if you ‘get lost’. The following recommendations will help:

1. Work on only one row at a time in any one step, unless the operation is quite straightforward and the line in question isn’t involved in any other procedures in that step.

2. Generally speaking, the first element to eliminate is the lower left-hand corner, then the element directly above it.

3. Work to eliminate all elements below the leading diagonal then roll back lower rows to eliminate elements above the leading diagonal.

**Using the Gaussian elimination method to find an inverse**

In the previous section we used an augmented matrix \([A \ | \ B]\) to find solutions for a system of linear equations.

The Gaussian elimination method can also be used to find the inverse of matrix \(A\). This is achieved by augmenting \(A\) with \(I\) — as in \([A \ | \ I]\) and performing row reduction procedures to produce \([I \ | A^{-1}]\). This is shown in the following example.
WORKED Example 3

Find the inverse of \( A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \).

**THINK**

1. Set up the augmented matrix \([A | I]\).

\[
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}
\]

2. Proceed with row reduction steps as for the previous examples.

Eliminate \( a_{21} \) first using \( R_2 - R_1 \).

3. Roll back row 2 to eliminate \( a_{12} \).

4. This augmented matrix is now in the form \([I | A^{-1}]\).

\[
\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}
\]

\[ A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \]

5. Check this is so by verifying \( A^{-1}A = I \).

\[
A^{-1}A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

WORKED Example 4

Find \( A^{-1} \) (if it exists) for \( A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \).

**THINK**

1. Set up the augmented matrix.

\[
\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{bmatrix}
\]

Eliminate \( a_{21} \).

\[
A^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}
\]

**WRITE**
**THINK**

2 Eliminate $a_{32}$.

R$_3 + \frac{1}{2}$R$_2$:
\[
\begin{pmatrix}
1 & 2 & 1 & 1 & 0 \\
0 & -2 & 0 & -1 & 1 \\
0 & 0 & 3 & -\frac{1}{2} & 1
\end{pmatrix}
\]

3 Eliminate $a_{12}$.

R$_1 + R_2$:
\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & -2 & 0 & -1 & 1 \\
0 & 0 & 3 & -\frac{1}{2} & 1
\end{pmatrix}
\]

4 Eliminate $a_{13}$.

R$_1 - \frac{1}{3}$R$_3$:
\[
\begin{pmatrix}
1 & 0 & 0 & \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\
0 & -2 & 0 & -1 & 1 \\
0 & 0 & 3 & -\frac{1}{2} & 1
\end{pmatrix}
\]

5 Reduce $a_{22}$ and $a_{33}$.

R$_2 + 2$ and R$_3 + 3$:
\[
\begin{pmatrix}
1 & 0 & 0 & \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\
0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 1 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3}
\end{pmatrix}
\]

This is now in the form $[IA^{-1}]$.

State $A^{-1}$.

\[
A^{-1} = \begin{pmatrix}
\frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\
\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{3}
\end{pmatrix}
\]

6 Verify that $A^{-1}A = I$.

\[
A^{-1}A = \begin{pmatrix}
\frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\
\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{3}
\end{pmatrix} \begin{pmatrix}
1 & 2 & 1 \\
1 & 0 & 1 \\
0 & 1 & 3
\end{pmatrix}
\]

\[
A^{-1}A = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
History of mathematics

CARL FRIEDRICH GAUSS (1777–1855)

During his lifetime...

Karl Marx and Friedrich Engels jointly publish ‘The Communist Party Manifesto’.

The Frenchman, Nicéphore Niepce produces the world’s first photographic image.

Samuel Morse develops the ‘Morse Code’.

You will come across the work of Carl Gauss in many fields of mathematics. His work is so diverse that he is considered by many to be the greatest mathematician of all times. Gauss was born in Brunswick, Germany, on April 20, 1777. From the age of three he had shown his superior skills by performing mental calculations and by the age of ten had progressed to algebra and analysis. While still a teenager he had developed the ‘least squares’ method used in statistical data, had devised a proof that a regular 17-sided polygon could be constructed using a compass and ruler and his quadratic reciprocity theorem. At the age of 22 he received his Ph.D, proving the Fundamental Theorem of Algebra. The next year he published his work on number theory, organising previous work and introducing the notion of modular arithmetic.

In 1801, he used the information from three sightings of an asteroid, Ceres, to calculate its orbit. In the process of this work he showed that the variation involved in experimental data followed a bell-shaped curve, now called the Gaussian or normal distribution. In 1807 Gauss became professor of astronomy and director of the new observatory in Göttingen. His work involved branches of astronomy, mechanics, optics, geodesy and magnetism, and in collaboration with Weber, the first practical telegraph. His extensive use of complex numbers advanced the acceptance of them by fellow mathematicians, although he was generally not supportive of young, aspiring scholars.

He died in Göttingen in 1855. His memorial bears the 17-point star of his early fame.

Questions
1. Try to reproduce the 17-point star with all angles and sides the same.
2. Research the Fundamental Theorem of Algebra.
3. What is geodesy?
4. Find out about the quadratic reciprocity theorem.

remember

For a set of simultaneous equations represented by matrices $A$, $X$, $B$ such that $AX = B$, Gaussian elimination can be used to change an augmented matrix $[A|X]$ to $[I|X]$ and $[A|I]$ to $[I|A^{-1}]$ where $I$ is the identity matrix.
EXERCISE 5B  
Solving simultaneous equations — the Gaussian elimination method

Use the Gaussian method of elimination to solve the following systems of linear equations.

1. \[ \begin{align*} 
2x - y &= 1 \\
3x + 2y &= 5 
\end{align*} \]

2. \[ \begin{align*} 
3a + 2b &= -11 \\
a + 3b &= -6 
\end{align*} \]

3. \[ \begin{align*} 
2y - z &= 5 \\
x + z &= -1 \\
2x - y - z &= 6 
\end{align*} \]

4. \[ \begin{align*} 
3x + 4y - z &= 11 \\
x - y + 2z &= -1 \\
5x + y - 3z &= 7 
\end{align*} \]

5. \[ \begin{align*} 
3x + 4y + z &= -10 \\
2x + y + 2z &= -5 \\
x - 2y + 2z &= 0 
\end{align*} \]

6. Use Gaussian elimination to find the inverses of the following matrices, if they exist.

\[ \begin{bmatrix} 
1 & 2 \\
2 & 3 
\end{bmatrix} \quad \begin{bmatrix} 
3 & 1 \\
2 & 2 
\end{bmatrix} \quad \begin{bmatrix} 
1 & 2 & 2 \\
0 & 1 & 3 \\
1 & 3 & 2 
\end{bmatrix} \quad \begin{bmatrix} 
2 & 1 & 0 \\
0 & 2 & 0 \\
2 & 2 & 1 
\end{bmatrix} \]

7. Write each of the systems of linear equations below in the matrix form \( AX = B \) and find inverses to solve the equations.

\[ \begin{align*} 
a & \quad \begin{align*} 
2x + y &= 6 \\
x + 3y &= 4 \frac{1}{2} 
\end{align*} \\
\end{align*} \]

\[ \begin{align*} 
b & \quad \begin{align*} 
3x + 2y &= 9 \\
x + 4y &= 7 \\
2x - z &= 5 
\end{align*} \\
\end{align*} \]

\[ \begin{align*} 
c & \quad \begin{align*} 
x + y - z &= -6 \\
2x - y + 2z &= 1 \\
x + 2y &= -7 
\end{align*} \\
\end{align*} \]

Introducing determinants

As mentioned in the previous section, Gaussian elimination is just one method used to solve systems of linear equations. Other methods involving determinants of matrices were used as early as 1100 BC by the Chinese and more recently by Gabriel Cramer (1704–52) and Augustine Cauchy (1789–1857).

You were introduced to determinants in chapter 3 on matrices where a quick test to determine whether a matrix was singular (that is, had no inverse) was to evaluate its determinant.

For example, for \( A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \) we have \( A^{-1} = \frac{1}{2 \times 1 - 2 \times 1} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \frac{1}{0} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \).

Because the determinant of \( A = 0 \), no inverse exists such that \( A^{-1}A = I \).

By definition, if \( A = [a] \), then \( \det A = |a| \) for a \( 1 \times 1 \) matrix.
Determinant of a $2 \times 2$ matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Therefore the determinant can only be found for a square matrix $A$ and is denoted by straight brackets, $\mid \mid$, not the square brackets [ ] used for a matrix. Its value is a single numerical answer, not a table of values like a matrix.

**WORKED Example 5**

Evaluate the determinant of $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

**THINK**

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad - bc$.

**WRITE**

\[
\begin{align*}
\det A &= 2 \times 2 - 3 \times 1 \\
\det A &= 4 - 3 \\
\det A &= 1
\end{align*}
\]

Determinant of a $3 \times 3$ matrix

As with the inverse of matrices, we need to be able to find determinants of matrices larger than $2 \times 2$.

The determinant of a $3 \times 3$ matrix involves evaluating three $2 \times 2$ determinants.

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\det A = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$

where the 3 sub-determinants are referred to as minors.

Therefore $\det A = a(ei - hf) - b(di - fg) + c(dh - eg)$

$= aei - afh - bdi + bfg + cdh - ceg$

Note that the coefficients of each sub-determinant are the elements of row 1 and the minor of $a_{11}$ is formed by using elements not in row 1 or column 1. That is, mentally cross out the first row and first column.

\[
\begin{bmatrix}
\dot{a} & b & c \\
\dot{d} & e & f \\
\dot{g} & h & i
\end{bmatrix}
\]

and similarly for the other minors.

The second coefficient is given a negative sign.
If \( \det A = 0 \) then the inverse of \( A \) does not exist since \( \det A = ad - bc \), where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). In this case \( A \) is said to be singular.

A special set of simultaneous equations such as
\[
ax + by = 0 \\
\]
\[
cx + dy = 0
\]
is said to be homogeneous, where all the right-hand side constants are zero. The trivial solution to this system yields \( x = y = 0 \).

If \( \det A = 0 \) then an infinite number of non-trivial solutions exists. However, if \( \det A \neq 0 \) only the trivial solutions exist. You will encounter questions later in this chapter that deal with this situation.

**Remember**

1. By definition, if \( A = [a] \), then \( \det A = |a| \) for a \( 1 \times 1 \) matrix.
2. If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) then \( \det A = a d - b c \).
3. If \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \), \( \det A = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \)

where the 3 sub-determinants are referred to as minors.
Chapter 5 Matrices and their applications

Introducing determinants

Evaluate the determinants of the following matrices:

\[ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 5 & 1 \\ 2 & 3 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 \\ \frac{1}{3} & 3 \end{vmatrix} \quad \begin{vmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & 2 \end{vmatrix} \]

Evaluate the following determinants:

\[ \begin{vmatrix} a & -3 & 2 \\ -1 & 4 & 3 \\ 0 & 2 & 5 \end{vmatrix} \quad \begin{vmatrix} -1 & -1 & 0 \\ -3 & 4 & 2 \\ 2 & 3 & 5 \end{vmatrix} \quad \begin{vmatrix} 0 & 2 & -3 \\ -3 & 1 & 2 \\ 6 & 3 & 2 \end{vmatrix} \]

\[ \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 0 & 3 & 4 \end{vmatrix} \quad \begin{vmatrix} 2 & 5 & -1 \\ 3 & -2 & 4 \\ -1 & -2 & -3 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 0 \\ 2 & 3 & -5 \\ 6 & 8 & 1 \end{vmatrix} \]

\[ \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} \quad \begin{vmatrix} 2 & 1 & 3 \\ 4 & 0 & 5 \\ 0 & 0 & 6 \end{vmatrix} \]

\[ \begin{vmatrix} 4 & 10 & -2 \\ 3 & -2 & 4 \\ -1 & -2 & -3 \end{vmatrix} \]

Properties of determinants

In question 5 of the previous exercise, review parts \( a \) and \( b, f, g \) and \( h \). Do you notice any patterns that you think could occur in other situations? In fact there are 8 properties of determinants that can be very useful. We have already used one of them (see property 6 given on the next page) in using the Gaussian method to solve systems of linear equations.

The 8 properties of determinants are given below.

**Property 1  Determinant of a transpose**

For every square matrix \( A \), \( \det A = \det A' \) where \( \det A' \) is the determinant that results from the transpose of \( A \) as seen in questions 5 \( a \) and \( b \) above.

**Property 2  Identical rows**

If 2 or more rows (or columns) of a matrix are identical or in proportion, then \( \det A = 0 \) (this can be seen in question 5 \( g \)).
Property 3  Zero rows/columns
If all the elements of a row (or column) are zero, then det \( A = 0 \) (see question 5 f).

Property 4  Interchanging rows or columns
If 2 rows (or columns) of \( A \) are interchanged to give \( B \), then det \( A = -\det B \) (see questions 5 h and i).

Property 5  Multiples of rows/columns
If a row (or column) of matrix \( A \) is multiplied by a constant \( k \) (where \( k \neq 0 \)), to give matrix \( B \), then det \( B = k \det A \) (see questions 5 e and j).

Property 6  Adding rows/columns
If a non-zero multiple of a row or column of \( A \) is added to another row or column, then the determinant is unchanged.

Property 7  Zero lower-triangular matrix
If all the elements below the leading diagonal are zero, then det \( A \) equals the product of the elements on the leading diagonal (see question 5 h).

Property 8  Det \( I \)
If \( I \) is the identity matrix then det \( I = 1 \) (this property follows from property 7).

Some of these properties are easier to identify than others. For instance, a matrix with a zero row or column is readily identified. As well as these properties there are other short-cut methods that are very convenient for calculating determinants. The following expansion is one of these.

Expansion of a 3 \( \times \) 3 determinant using any row or column
The initial example of expansion of a 3 \( \times \) 3 determinant (worked example 6 on page 200) used the first row as coefficients of the minors. However, any row or column can be used — with the following adaptations.
1. A minor can be obtained by blocking out all the elements of the row and column of the coefficient.
2. Alternating signs are attached to the coefficients of each minor, as shown:

\[
\begin{array}{ccc}
+ & - & + \\
- & + & - \\
+ & - & + \\
\end{array}
\]

*Note:* The initial example (worked example 6) used a \((-)\) in front of the second coefficient because the first row elements were used as the coefficients of expansion.
If the elements of the second row had been used as the coefficients of expansion, then the signs on the minors would have been $-$, $+$, $-$. The same would have happened if the second column had been used.

Follow the next worked example to see how this alternative row or column can be used.

WORKED Example 7

Evaluate

\[
\begin{vmatrix}
1 & 2 & 1 \\
1 & 3 & 6 \\
1 & 4 & 9
\end{vmatrix}
\]

THINK

1. Because we can expand by any row or column, use column 1 as it will have coefficients of 1.
   (i) With a different colour pen write in the signs of the elements. Don’t get them confused and think $a_{21} = -1$.
   (ii) Draw an arrow to indicate the row/column of expansion.

WRITE

\[
\begin{vmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & +
\end{vmatrix}
\]

= 1(27) - 1(18) + 1(12)
= 3 - 14 + 9
= -2

In the example above any row or column could have been used, but you can see that if column 1 is used the coefficients will be 1 — a number that is easy to multiply by. Also, any row or column with mostly zeros allows you to complete the expansion faster, so it is useful to use that row or column to expand by.

For square determinants greater than $3 \times 3$, this simple alternating pattern of signs is continued.

For example a $4 \times 4$ determinant can be evaluated using alternating signs of the coefficients. The signs in front of the coefficients can be determined from the matrix shown below.

\[
\begin{vmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & +
\end{vmatrix}
\]
remember

All determinants have the following properties:
1. For every square matrix \( A \), \( \det A = \det A' \) where \( A' \) is the determinant that results from the transpose of \( A \).
2. If 2 or more rows (or columns) of a matrix are identical or in proportion, then \( \det A = 0 \).
3. If all the elements of a row (or column) are zero, then \( \det A = 0 \).
4. If 2 rows (or columns) of \( A \) are interchanged to give \( B \), then \( \det A = -\det B \).
5. If a row (or column) of matrix \( A \) is multiplied by a constant \( k \) (where \( k \neq 0 \)), to give matrix \( B \), then \( \det B = k \det A \).
6. If a non-zero multiple of a row or column of \( A \) is added to another row or column, then the determinant is unchanged.
7. If all the elements below the leading diagonal are zero, then \( \det A \) equals the product of the elements on the leading diagonal.
8. If \( I \) = identity matrix then \( \det I = 1 \).

EXERCISE 5D

Properties of determinants

Evaluate the following determinants:

1 a
\[
\begin{vmatrix}
1 & 2 \\
4 & 5
\end{vmatrix}
\]

b
\[
\begin{vmatrix}
1 & 4 \\
2 & 5
\end{vmatrix}
\]

c State the property involved.

2 a
\[
\begin{vmatrix}
1 & 2 & 1 \\
3 & 6 & 0 \\
1 & 2 & 1
\end{vmatrix}
\]

b State the property involved.

3 a
\[
\begin{vmatrix}
0 & 0 & 4 \\
2 & 0 & 5 \\
3 & 0 & -2
\end{vmatrix}
\]

b State the property involved.

4 a
\[
\begin{vmatrix}
2 & 3 & 6 \\
1 & -1 & 1 \\
2 & 1 & 0
\end{vmatrix}
\]

b
\[
\begin{vmatrix}
3 & 2 & 6 \\
-1 & 1 & 1 \\
1 & 2 & 0
\end{vmatrix}
\]

c State the property involved.

5 a
\[
\begin{vmatrix}
2 & 5 \\
4 & 1
\end{vmatrix}
\]

b
\[
\begin{vmatrix}
4 & 10 \\
4 & 1
\end{vmatrix}
\]

c State the property involved.

6 a
\[
\begin{vmatrix}
2 & 4 \\
0 & 1
\end{vmatrix}
\]

b
\[
\begin{vmatrix}
2 & 4 \\
2 & 5
\end{vmatrix}
\]

c State the property involved.

7 a
\[
\begin{vmatrix}
2 & -1 & 3 \\
0 & 4 & 7 \\
0 & 0 & 1
\end{vmatrix}
\]

b State the property involved.
8 a \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

b State the property involved.

9 Use any of the properties investigated in earlier sections to evaluate the following determinants:

\[
\begin{align*}
a & \begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} & b & \begin{bmatrix} 4 & 3 & 1 \\ -1 & 6 & 1 \\ 2 & 5 & 1 \end{bmatrix} & c & \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \\
d & \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 4 \\ 3 & 2 & -1 \end{bmatrix} & e & \begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 3 \end{bmatrix}
\end{align*}
\]

Inverse of a 3 × 3 matrix

In chapter 3 on matrices you were introduced to the idea of an inverse matrix \(A^{-1}\) where \(AA^{-1} = I\).

For \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) the inverse matrix \(A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\), where \(ad - bc \neq 0\) and \(ad - bc = \text{det} A\).

This rule is limited to 2 × 2 matrices. However, in its most general form it can be used to find the inverse of any square matrix, if the inverse exists.

The steps below demonstrate how the above formula can be used to find the inverse of a 3 × 3 matrix.

For matrix \(A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 5 \\ 9 & 7 & 8 \end{bmatrix}\)

1. Matrix \(C\), the cofactor matrix of \(A\), is obtained by replacing each element of \(A\) with its corresponding cofactor or signed minor.

\[
C = \begin{bmatrix}
6 & 5 & -4 & 5 & 4 & 6 \\
7 & 8 & 9 & 8 & 9 & 7 \\
3 & 1 & 2 & 1 & 2 & 3 \\
7 & 8 & 9 & 8 & 9 & 7 \\
3 & 1 & 2 & 1 & 2 & 3 \\
6 & 5 & 4 & 5 & 4 & 6
\end{bmatrix}
\]

= \begin{bmatrix}
13 & 13 & -26 \\
-17 & 7 & 13 \\
9 & -6 & 0
\end{bmatrix}

\]
2. The adjoint of $A$ is the transpose of the cofactor matrix and is written $\text{adj} A$.

$$\text{adj} A = \begin{bmatrix} 13 & -17 & 9 \\ 13 & 7 & -6 \\ -26 & 13 & 0 \end{bmatrix}$$

The adjoint matrix has the property that $A(\text{adj} A) = (\text{det} A)I$ and since $\text{det} A$ is a scalar we can divide by $\text{det} A$ to produce

$$\frac{A \text{adj} A}{\text{det} A} = I$$

therefore

$$\frac{\text{adj} A}{\text{det} A} = A^{-1}.$$ 

You can verify this result by showing $A^{-1}A = I$.

In its simplest form, this is the formula that was used to find the inverse of $2 \times 2$ matrices.

### WORKED Example 8

Use the cofactor/adjoint matrices to find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$.

**THINK**

1. Set up the cofactor matrix using signed minors and evaluate.

$$C = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 3 & 0 & 3 & 0 & 1 \\ -1 & 2 & 1 & 2 & 1 & -1 \\ 1 & 3 & 0 & 3 & 0 & 1 \\ -1 & 2 & 1 & 2 & 1 & -1 \\ 1 & 2 & 0 & 2 & 0 & 1 \end{bmatrix}$$

2. Write the adjoint as the transpose of $C$.

$$\text{adj} A = \begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$
THINK

3 Calculate det $A$ and set up $A^{-1}$.

$$\text{det } A = 1(3 - 2) = 1 \text{ (down column 1)}$$

$$A^{-1} = \frac{\text{adj } A}{\text{det } A}$$

$$A^{-1} = \begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

4 Verify this matrix is $A^{-1}$ by testing $AA^{-1} = I$.

$$A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This method can now be used to solve systems of linear equations involving a $3 \times 3$ matrix.

WORKED Example 9

Solve the system of linear equations given below.

$$x - y - z = 0$$
$$2x + y = 4$$
$$x + y + z = 2$$

THINK

1 Set up the matrices in the form $AX = B$.

$$AX = B$$

where $A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

WRITE

2 Rearrange to change $X$ to be the subject.

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

Continued over page
**THINK**

3 Use the cofactor matrix to find $A^{-1}$.

\[
C = \begin{bmatrix}
1 & 0 & -2 \\
1 & 1 & 2 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & -2 & 1 \\
0 & 2 & -2 \\
1 & -2 & 3
\end{bmatrix}
\]

4 Find the adjoint and det $A$.

\[
\text{adj } A = \begin{bmatrix}
1 & 0 & 1 \\
-2 & 2 & -2 \\
1 & -2 & 3
\end{bmatrix}
\]

\[
det A = 1(0 - -1) - 1(0 - -2) + 1(1 - -2)
\]

\[
= 1 - 2 + 3
\]

\[
= 2
\]

5 Find the inverse.

\[
A^{-1} = \frac{\text{adj } A}{\text{det } A}
\]

\[
= \frac{1}{2} \begin{bmatrix}
1 & 0 & 1 \\
-2 & 2 & -2 \\
1 & -2 & 3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
-1 & 1 & -1 \\
\frac{1}{2} & -1 & \frac{1}{2}
\end{bmatrix}
\]

6 Solve for $X$ and check the values provided.

\[
X = \begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} \\
-1 & 1 & -1 \\
\frac{1}{2} & -1 & \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
0 \\
4 \\
2
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}
\]

So $x = 1, y = 2, z = -1$. 
## Inverse of a 3 × 3 matrix

### Example 8

1. Find the determinant, ii) cofactor matrix iii) adjoint matrix and iv) inverse of each of the following.
   
   - **a)** \[
   \begin{bmatrix}
   1 & 1 \\
   2 & 3
   \end{bmatrix}
   \]
   - **b)** \[
   \begin{bmatrix}
   2 & 5 \\
   6 & 9
   \end{bmatrix}
   \]
   - **c)** \[
   \begin{bmatrix}
   1 & 2 & 1 \\
   2 & 0 & 4 \\
   3 & -2 & 5
   \end{bmatrix}
   \]
   - **d)** \[
   \begin{bmatrix}
   -1 & 4 & 0 \\
   2 & 3 & -1 \\
   -1 & 4 & 3
   \end{bmatrix}
   \]

2. Find the inverse of the following matrices, if they exist.
   
   - **a)** \[
   \begin{bmatrix}
   2 & 1 & 1 \\
   3 & -3 & 2 \\
   2 & 0 & 1
   \end{bmatrix}
   \]
   - **b)** \[
   \begin{bmatrix}
   3 & 5 & -4 \\
   2 & -1 & 3 \\
   3 & 2 & -2
   \end{bmatrix}
   \]
   - **c)** \[
   \begin{bmatrix}
   -1 & 2 & 2 \\
   -3 & 6 & 1 \\
   2 & 0 & 5
   \end{bmatrix}
   \]

3. Find the inverse of \[
   \begin{bmatrix}
   1 & 2 & -1 \\
   0 & 3 & 1 \\
   2 & 0 & -2
   \end{bmatrix}
   \] and use it to solve the system of linear equations given by:
   
   \[
   \begin{align*}
   x + 2y - z &= 0 \\
   3y + z &= 9 \\
   2x - 2z &= 8
   \end{align*}
   \]

4. Solve the following systems of linear equations.
   
   - **a)** \[
   \begin{align*}
   x + y - z &= 9 \\
   -2x - y + z &= -11 \\
   x + 2y + 2z &= 0
   \end{align*}
   \]
   - **b)** \[
   \begin{align*}
   x - y - 4z &= -11 \\
   6x + 2y + z &= 9 \\
   -3x - 2y + 2z &= 3
   \end{align*}
   \]

5. Find \(x\) if \[
   \begin{vmatrix}
   2 + x & 1 \\
   1 & 2 - x
   \end{vmatrix} = 0.
   \]
Cramer’s Rule for solving linear equations

Determinants on their own can also be used to solve systems of linear equations. In fact, determinants were first studied in this context. This method of solving linear equations is known as Cramer’s Rule and will be used to solve systems of 2 linear equations.

For the equations

\[ ax + by = u \]

and

\[ cx + dy = v \]

\[ x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{provided} \quad ad - bc \neq 0. \]

Note the pattern of elements in the determinant of the numerators. Mathematics isn’t referred to as the study of patterns without good reason.

The proof of this statement follows.

Let the system of linear equations be

\[ ax + by = u \]

and

\[ cx + dy = v. \]

Written in matrix form they appear as

\[ \begin{bmatrix} a & b \\
                c & d \end{bmatrix} \begin{bmatrix} x \\
                               y \end{bmatrix} = \begin{bmatrix} u \\
                                   v \end{bmatrix}. \]

In matrix equation form, this is

\[ AX = B \]

\[ X = A^{-1}B. \]

\[ \begin{bmatrix} x \\
                  y \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\
                                                             -c & a \end{bmatrix} \begin{bmatrix} u \\
                                                                       v \end{bmatrix} \]

\[ = \frac{1}{ad - bc} \begin{bmatrix} du - bv \\
                                        -cu + av \end{bmatrix} \]

Therefore, \( x = \frac{du - bv}{ad - bc} \) and \( y = \frac{-cu + av}{ad - bc} \) where \( ad - bc \neq 0. \)

The numerators and denominators of both these expressions can be written as determinants:

\[ x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \]
Use Cramer's Rule to solve the following linear equations.

\[2x + 2y = 3\]
\[x - y = \frac{1}{2}\]

**THINK**

1. Write the general equations in matrix form and apply Cramer's Rule.

   For \[\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}\]

   \[x = \frac{u}{\det} \quad \text{and} \quad y = \frac{v}{\det}\]

2. Substitute the given values for \(a, b, c, d\) and \(u, v\).

\[\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{1}{2} \end{bmatrix}\]

   \[x = \frac{3 \cdot \frac{1}{2} - 2 \cdot 1}{2 \cdot (-1) - 1 \cdot 2} = \frac{-4}{4} = 1\]

   \[y = \frac{2 \cdot 1 - 3 \cdot (-1)}{2 \cdot (-1) - 1 \cdot 2} = \frac{5}{4} - \frac{3}{2} = \frac{1}{2}\]

3. Verify these results by substitution into the original system of equations

\[2(1) + 2\left(\frac{1}{2}\right) = 3 \quad \text{(Verified for the 1st equation)}\]
\[1 - \frac{1}{2} = \frac{1}{2} \quad \text{(Verified for the 2nd equation)}\]

Cramer's Rule can readily be extended to find the solution of 3 equations in 3 unknowns.

**WORKED Example 11**

Solve: \[2x + y + z = 3\]
\[x + 2y - z = -6\]
\[5x - 2z = -1.\]

**THINK**

1. Write the system in matrix form.

\[A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -6 \\ -1 \end{bmatrix}\]
THINK

Matrix $B$ is used as column 1 for $x$, column 2 for $y$ and column 3 for $z$.

WRITE

\[
\begin{align*}
x &= \begin{vmatrix} 3 & 1 & 1 \\ -6 & 2 & -1 \\ -1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -6 & -1 \\ 5 & -1 & -2 \end{vmatrix} \\
y &= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 0 & -2 \end{vmatrix} \\
z &= \begin{vmatrix} 2 & 1 & 3 \\ 1 & 2 & -6 \\ 5 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 5 & 0 & -2 \end{vmatrix}
\end{align*}
\]

1. Evaluate the determinants, don’t forget the signed minors.

\[
x = \frac{-21}{-21} = 1
\]

\[
y = \frac{42}{-21} = -2
\]

\[
z = \frac{-63}{-21} = 3
\]

2. Verify these results by substituting into the 3 equations.

\[
2(1) + -2 + 3 = 3 \quad \text{(Verifying the 1st equation)}
\]

\[
1 + 2(-2) - 3 = -6 \quad \text{(Verifying the 2nd equation)}
\]

\[
5(1) - 2(3) = -1 \quad \text{(Verifying the 3rd equation)}
\]

Remember

\[
\text{Cramer’s Rule for solving linear equations: } ax + by = u \\
\text{\hspace{1cm}} cx + dy = v
\]

\[
\text{states that } \quad x = \frac{u \cdot d - b \cdot v}{a \cdot d - b \cdot c} \quad \text{and } \quad y = \frac{a \cdot v - c \cdot u}{a \cdot d - b \cdot c}
\]
Cramer’s Rule for solving linear equations

Use Cramer’s Rule to solve the following:

1. \[ 2x + y = 1 \]
   \[ 3x - 2y = 12 \]

2. \[ 2x + 4y = 14 \]
   \[ 3x - y = -7 \]

3. \[ -x - y = 7 \]
   \[ 4x - y = -3 \]

4. \[ 3x + 2y = 6 \]
   \[ -2x + y = 3 \]

5. \[ x + y + z = 4 \]
   \[ 2x + y - z = 3 \]
   \[ 3x + 3y + 2z = 10 \]

6. \[ -x + 2y + 5z = -1 \]
   \[ -2x + 3y - z = 7 \]
   \[ x - 2y - 2z = 0 \]

7. Evaluate the following:
   \[ \begin{vmatrix} x & y & z \\ u & 1 & 1 \\ v & 1 & 1 \end{vmatrix} \quad \begin{vmatrix} 1 & x & \chi \\ y & 1 & x \end{vmatrix} \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 - x & 1 \end{vmatrix} \quad \begin{vmatrix} b & a & a \\ a & b & a \\ a & a & b \end{vmatrix} \]

8. Solve each of the following equations:
   \[ \begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 0 \]
   \[ \begin{vmatrix} 1 & a & a \\ a & 1 & 0 \end{vmatrix} = 0 \]
   \[ \begin{vmatrix} a & a \\ 2 & a - 1 \end{vmatrix} = 0 \]
   \[ \begin{vmatrix} a & 0 & 0 \\ 0 & a - 1 & 0 \\ 0 & 0 & a - 2 \end{vmatrix} = 0 \]
   \[ \begin{vmatrix} x - 1 & -3 \\ 2 & x - 6 \end{vmatrix} = 0 \]
   \[ \begin{vmatrix} x - 3 & 0 & 0 \\ 0 & x & 3 \\ 0 & 1 & x - 2 \end{vmatrix} = 0 \]

9. If \[ A = \begin{bmatrix} 3 & 0 & 5 \\ 2 & 4 & 6 \\ 1 & 2 & 4 \end{bmatrix} \] find \( A^{-1} \).

10. \[ A = \begin{bmatrix} 2 & 1 & -1 \\ 7 & -9 & 3 \\ 2 & 4 & 5 \end{bmatrix} \]
    a. Find \( \det A \).
    b. Find \( A^{-1} \).
    c. Use the result from b to solve the system:
        \[ 2x + y - z = 2 \]
        \[ 7x - 9y + 3z = -5 \frac{1}{2} \]
        \[ 2x + 4y + 5z = 5 \]
11 Show that \[
\begin{bmatrix}
1 & a & 0 \\
0 & 1 & 0 \\
0 & b & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & -a & 0 \\
0 & 1 & 0 \\
0 & -b & 1
\end{bmatrix}.
\]

12 If \( A^{-1} = \begin{bmatrix}
1 & 2 & 1 \\
0 & 1 & 2 \\
1 & 0 & 1
\end{bmatrix} \)

\( a \) find \( A \) and \( A' \)

\( b \) verify that \((A')^{-1} = (A^{-1})'\).

13 Evaluate for complex \( i \)

\( a \) \[
\begin{bmatrix}
1 & i \\
-1 & 0
\end{bmatrix}
\]

\( b \) \[
\begin{bmatrix}
1 & i & 0 \\
-i & i & 1 \\
1 & i & 0
\end{bmatrix}
\]

\( c \) \[
\begin{bmatrix}
1 - i & i & 1 + i \\
1 + i & 1 - i & i \\
i & 1 + i & 1 - i
\end{bmatrix}
\]

14 Verify that \( \det(AB) = \det A \times \det B \) for the following:

\( a \) \[
A = \begin{bmatrix}
1 & 2 & 1 \\
0 & 2 & 4 \\
4 & 3 & 1
\end{bmatrix}
\]

\( B = \begin{bmatrix}
2 & 1 & 0 \\
3 & 2 & -1 \\
1 & 4 & 5
\end{bmatrix}
\]

\( b \) \[
A = \begin{bmatrix}
5 & 1 & -1 \\
1 & -3 & 2 \\
2 & 0 & 1
\end{bmatrix}
\]

\( B = \begin{bmatrix}
3 & 0 & 2 \\
4 & 1 & -2 \\
0 & -1 & 1
\end{bmatrix}
\]

---

**Graphics Calculator tip!**

**Matrix operations**

Pen-and-paper methods of computing inverses, solving systems of linear equations and finding determinants are tedious. So you will now appreciate the convenience of being able to use a graphics calculator to provide quick and reliable answers to some of the problems that you have encountered in this chapter.

**Entering a matrix**

Use \( A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix} \)

(a) Press \( \text{ON} \ \text{CLEAR} \ \text{MATRIX} \). Select \( \text{EDIT} \) and press \( \text{ENTER} \) then choose \( 1: [A] \) and press \( \text{ENTER} \). To enter a \( 3 \times 3 \) matrix, type \( 3 \ \text{ENTER} \ 3 \ \text{ENTER} \) for the dimensions of matrix \( A \).

(b) Notice that the cursor highlights element \( a_{11} \) and the coordinate at the bottom left of the screen will display \( 1, 1 = \). Enter each element as the cursor moves across the row.
(c) Continue with row 2. Note the position of the cursor and coordinate.

(d) When you have completed entering all the elements press \( \text{2nd} \ \text{MODE} \) to quit.

(e) Enter matrix \( B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -4 & 5 \\ 2 & 4 & 0 \end{bmatrix} \) by pressing \( \text{MATRIX} \) and selecting \( \text{EDIT} \). Then press the ‘down’ arrow to select \( 2: [B] \) and press \( \text{ENTER} \). Change the dimensions to a \( 3 \times 3 \) matrix as before and enter the elements of \( B \) as listed. Note that you can move around inside the matrix using the cursor keys. When you have finished type \( \text{2nd} \ \text{MODE} \) (to quit).

**Operations with matrices**

(i) Addition: \( A + B \)

Press \( \text{MATRIX} \ \text{ENTER} \) and select \( 1: [A] \) \( + \) then press \( \text{MATRIX} \) and use the arrow down to highlight \( 2: [B] \). Press \( \text{ENTER} \) to display the sum.

(ii) Repeat the procedure for \( A - B \) and \( A \times B \).

**To find the inverse of \( B \)**

Press \( \text{MATRIX} \) select \( 2: [B] \) and press \( \text{ENTER} \); then press \( x^{-1} \) \( \text{ENTER} \). To multiply this answer \( (B^{-1}) \) by \( B \) press \( \times \) \( \text{MATRIX} \), select \( 2: [B] \) and press \( \text{ENTER} \) to display \( I \) as expected.

**Powers of \( A \)**

Press \( \text{CLEAR} \ \text{MATRIX} \) choose \( 1: [A] \) \( \text{ENTER} \) \( x^2 \) \( \text{ENTER} \) or use \( \text{^} \) \( 2 \). This \( \text{^} \) can be used for all powers from 0 to 255. Try \( A^0 \)!!

**Determinant of \( A \)**

Press \( \text{CLEAR} \ \text{MATRIX} \) select \( \text{MATH} 1: \text{det (} \) and press \( \text{ENTER} \). Then press \( \text{MATRIX} \), select \( 2: [B] \) and \( 1 \) to close the brackets and press \( \text{ENTER} \). The number 20 will be displayed.

---

**Solving simultaneous equations**

Consider \( x + y + z = 4 \)
\( 2x - y - 2z = 6 \)
\( 3x - 2y = 2 \)

which can be written in matrix form as \( C = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & -1 & -2 & 6 \\ 3 & -2 & 0 & 2 \end{bmatrix} \)

Enter the matrix \( C \) with dimensions \( 3 \times 4 \) (3 rows and 4 columns) use \( \text{-} \) key for negative values and press \( \text{2nd} \ \text{MODE} \) to quit. Press \( \text{MATRIX} \) select \( \text{MATH} \) and arrow down to \( \text{B: rref (} \) for reduced row echelon form (the screen will scroll down) and press \( \text{ENTER} \). Press \( \text{MATRIX} \) and select \( 3: [C] \), press \( 1 \) to close the brackets and press \( \text{ENTER} \). The display shows \( [I \mid X] \) so \( x = 2.72, y = 3.09 \) and \( z = -1.81 \).

Now you can rework a selection of the exercises from earlier sections to practise with this great tool.
Applications of determinants

Using determinants to find the equation of a line

One of the many applications of determinants is in determining the equation of a straight line.
You are familiar with the equation of a straight line through two points. Assume that the given line passes through points \( A(x_1, y_1) \) and \( B(x_2, y_2) \). Let \( P(x, y) \) be a point on this line.
As \( P \) is a point on the line then the slope of \( AP \) equals the slope of \( AB \).

\[
\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}
\]

Therefore \((y - y_1)(x_2 - x_1) = (y_2 - y_1)(x - x_1)\)

\[
yx_2 - yx_1 - y_1x_2 + x_1y_1 = y_2x - x_1y_2 - xy_1 + x_1y_1
\]

\((x_1y_2 - x_2y_1) - (xy_2 - x_2y) + (xy_1 - x_1y) = 0\)

\[
1 \times \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} - 1 \times \begin{vmatrix} x & x_2 \\ y & y_2 \end{vmatrix} + 1 \times \begin{vmatrix} x & x_1 \\ y & y_1 \end{vmatrix} = 0
\]

The multipliers of one have been included to display the determinant form more clearly and can be written as:

\[
\begin{vmatrix} 1 & 1 & 1 \\ x & x_1 & x_2 \\ y & y_1 & y_2 \end{vmatrix} = 0
\]

1 Find the equation of the straight line joining points \((2, 4)\) and \((4, -6)\) using determinants.

Using determinants to find the area of a triangle

Similarly the area of a triangle can be represented in a determinant form:
where the area of \( \triangle ABC \) equals the total area of the two trapeziums ADEB and BEFC less the area of trapezium ADFC.

\[
\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}
\]

2 Demonstrate that the area of \( \triangle ABC = \frac{1}{2} \)

and use this determinant to find the area of a triangle of your design.
Chapter 5 Matrices and their applications

Gaussian elimination method

- An augmented matrix \([A \mid B]\) can be used to find the solution of a set of simultaneous equations when Gaussian row reduction changes \([A \mid B]\) to \([I \mid X]\).
- An augmented matrix \([A \mid B]\) can be used to find the inverse of \(A \) when Gaussian row reduction changes \([A \mid B]\) to \([A^{-1} \mid I]\).

Determinants

- The determinant of \(A\) is written \(\det A\); \(\det A = ad - bc\) where \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\).
- The determinant of a \(3 \times 3\) matrix is found by using the alternating signs of attached to the coefficients of each minor.

| + - + | attached to the coefficients of each minor.
| - + - |
| + - + |

Adjoint matrix

- \(A^{-1} = \frac{\text{adj } A}{\det A}\), where \(\text{adj } A\) is the transpose of the cofactor matrix of \(A\), made up of the minors of \(A\).

Cramer’s Rule for solving linear equations

- Cramer’s Rule for solving equations: \(ax + by = u\) and \(cx + dy = v\)

\[
\begin{vmatrix} u & b \\ v & d \end{vmatrix}
\quad \text{and} \quad
\begin{vmatrix} a & u \\ c & v \end{vmatrix}
\]

states that \(x = \frac{u \cdot d - v \cdot b}{a \cdot d - c \cdot b}\) and \(y = \frac{a \cdot v - c \cdot u}{a \cdot d - c \cdot b}\), provided \(ad - bc \neq 0\).
1 Use the Gaussian method of elimination to solve the following system of linear equations:
   \[2x - y + z = -1\]
   \[3x + 2y - z = 4\]
   \[x - 2y + 2z = 1\]

2 Use Gaussian elimination to find the inverse of the following matrices:
   \[
   \begin{bmatrix}
   3 & 2 \\
   -1 & 1
   \end{bmatrix}
   \quad \text{and} \quad
   \begin{bmatrix}
   3 & 0 & 1 \\
   2 & 0 & 1 \\
   -1 & 1 & 2
   \end{bmatrix}
   \]

3 Evaluate the following determinants:
   \[
   \begin{vmatrix}
   1 & 2 & 3 \\
   1 & 1 & 2 \\
   -2 & 0 & 0
   \end{vmatrix}
   \quad \text{and} \quad
   \begin{vmatrix}
   2 & 3 & 2 \\
   0 & 1 & 4 \\
   2 & 1 & -1
   \end{vmatrix}
   \quad \text{and} \quad
   \begin{vmatrix}
   1 & 0 & 1 \\
   0 & 1 & 1 \\
   1 & 3 & 2
   \end{vmatrix}
   \]

4 Find the i determinant, ii cofactor matrix, iii adjoint matrix and iv inverse of the following:
   \[
   \begin{bmatrix}
   1 & 2 \\
   3 & 4
   \end{bmatrix}
   \quad \text{and} \quad
   \begin{bmatrix}
   3 & 1 & 1 \\
   2 & 1 & 2 \\
   -1 & 2 & -3
   \end{bmatrix}
   \]

5 Use the cofactor–adjoint method to solve the following system of linear equations:
   \[2x - y + 3z = 4\]
   \[-x + 2y - z = -5\]
   \[4x + y - 2z = 0\]

6 a State Cramer’s Rule for solving two equations in two unknowns.
   b Use this rule to solve
   \[2x - 3y = 7\]
   \[3x + y = 5\]