Recall that a function is important because it relates two variables to one another. To explore these relationships, we will make connections between the algebraic and graphical representations of these functions. Each representation provides us with essential information about the function. Sometimes the behaviour of a function (for example tidal heights over time) is most easily seen by looking at a sketch of its graph. The sketch must show the general shape of the function over its domain and show important features of the function. These may include x-intercepts, y-intercepts, endpoints, asymptotes, turning points and points of inflection, depending on the function being considered.

This chapter reviews much of the material on transformations covered in *MathsWorld Mathematical Methods Units 1 & 2*. Chapter 3 gives this material a more formal treatment.
### Power and polynomial functions

#### Power functions

Any function with rule in the form \( y = x^n \), where \( n \) is a constant, is called a **power function**. Initially, we will consider power functions and their graphs where \( n \) is a positive integer and then extend this to include \( n = \frac{1}{2}, -1, -2 \) and others. We will start by recalling the shapes of the basic graphs of power functions and some transformations of these.

The power function with \( n = 2 \) has rule \( y = x^2 \). This is a **quadratic function** and its graph is a **parabola**. A parabola is a symmetrical curve with its axis of symmetry being a vertical line that passes through the turning point.

For the graph of \( y = x^2 \), the axis of symmetry is the \( y \)-axis (which is the line \( x = 0 \)) and the turning point is at the origin \((0, 0)\). The domain of this function is the set of real numbers and the range is the set of values greater than or equal to zero, so \( x \in \mathbb{R}, y \in \mathbb{R}^+ \cup \{0\} \).

In general, when we refer to the domain of a function, we are referring to ‘the maximal domain’ or ‘the implied domain’ of the function, i.e. the largest allowable domain for that function.

For the graph of \( y = -x^2 \), the \( y \) values are the negative of those for \( y = x^2 \), and so the graph of \( y = -x^2 \) is the reflection in the \( x \)-axis of the graph of \( y = x^2 \). That is, it is an upside down parabola.

The power function with \( n = 3 \) has rule \( y = x^3 \). This is also an example of a **cubic function**, which is any polynomial function of degree 3. The graph of \( y = x^3 \) is a particular example of a cubic (polynomial) graph in which there is a **stationary point of inflection** at the origin \((0, 0)\). This means that for negative values of \( x \) the curve is turning one way, but for positive values of \( x \) it is turning the other way. At the origin, the gradient is zero (there is a stationary point). The domain and the range of this function are both the set of real numbers.

#### \( n = 2 \)

The graph of \( y = (x + b)^2 + c \) is an upright (positive) parabola with its turning point at \((-b, c)\).

The graph of \( y = -(x + b)^2 + c \) is an inverted parabola with its turning point at \((-b, c)\).

#### \( n = 3 \)

The graph of \( y = (x + b)^3 + c \) is a special case of a positive cubic graph with a stationary point of inflection at \((-b, c)\).

The graph of \( y = -(x + b)^3 + c \) is an inverted cubic graph with a stationary point of inflection at \((-b, c)\).
The power function with $n = 4$ has rule $y = x^4$. This is also an example of a **quartic function**, which is any polynomial function of degree 4. The graph of $y = x^4$ is a particular example of a **quartic graph**.

For the graph of $y = x^4$, the axis of symmetry is the $y$-axis (which is the line $x = 0$) and the turning point is at the origin $(0, 0)$. The domain of this function is the set of real numbers and the range is the set of real numbers greater than or equal to zero, so $x \in \mathbb{R}$, $y \in \mathbb{R}^+ \cup \{0\}$.

Compared to a parabola, a quartic graph is flatter near the turning point but rises more steeply the further away we get. Both graphs are shown on the axes at right. Note that the quartic graph is below the parabola ($x^4 < x^2$) when $x$ is between $-1$ and $1$, intersects the parabola at $x = \pm 1$, and is above the parabola ($x^4 > x^2$) for all other values of $x$.

### Graphs of power functions with odd and even powers

#### $y = x^n$, $n$ odd

If $n > 1$ and $n$ is odd, the graph of $y = x^n$ has a similar shape to that of $y = x^3$. There is a point of inflection at the origin $(0, 0)$. As $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to -\infty$.

The graph of $y = x^5$ contains the points $(-2, -32)$ and $(2, 32)$, whereas the graph of $y = x^3$ contains the points $(-2, -8)$ and $(2, 8)$. Both graphs pass through the points $(-1, -1)$ and $(1, 1)$ as well as $(0, 0)$.

#### $y = x^n$, $n$ even

We have seen that the quartic graph $y = x^4$ has a similar shape to a basic parabola. If $n$ is an even number, the graph of $y = x^n$ has a similar shape to that of $y = x^2$. There is a turning point at the origin $(0, 0)$. As $x \to \infty$, $y \to \infty$; and as $x \to -\infty$, $y \to \infty$.

The graph of $y = x^6$ contains the points $(-2, 64)$ and $(2, 64)$, whereas the graph of $y = x^4$ contains the points $(-2, 16)$ and $(2, 16)$. Both graphs pass through the points $(-1, 1)$ and $(1, 1)$ as well as $(0, 0)$. 

---
Power functions $y = x^n$, where $n = \frac{1}{2}, -1, -2$

The graphs of power functions we have met so far have $n \in \mathbb{N}$ and domain $\mathbb{R}$. The graphs of power functions with $n = \frac{1}{2}, -1$ and $-2$ have restricted domains.

The power function $y = x^{1/2}$ and its graph

The function $y = x^{1/2}$ is the same as $y = \sqrt{x}$ and is called the square root function. We know that if any real number is squared, the result is a number greater than or equal to zero. Therefore, we can only take the square root of a number greater than or equal to zero.

It follows that the domain of $y = \sqrt{x}$ is $\{x : x \geq 0\}$, i.e. $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$. As $y$ equals the positive square root of $x$, the range is $\{y : y \geq 0\}$, i.e. $\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$.

The graph of the square root function is called the square root graph. The square root graph is shaped like part of a parabola on its side. It is actually the reflection of the graph of $y = x^2$, $x \geq 0$ in the line $y = x$.

The graph of $y = x^{1/2}$ is drawn with a solid circle at the origin to indicate the end point of the graph. (An open circle is used for an endpoint that is not included.)

The power functions $y = x^{-1}$ and $y = x^{-2}$ and their graphs

The equation $y = x^{-1}$ is the same as $y = \frac{1}{x}$ and its graph is called a hyperbola. We know that $x$ cannot be equal to zero as $\frac{1}{x}$ is undefined. It follows that the maximum domain of $y = x^{-1}$ is the set of all real numbers except zero. It also follows that $y$ can take any value other than zero. Thus, both the domain and the range are $\mathbb{R} \setminus \{0\}$ or $(-\infty, 0) \cup (0, \infty)$. 

For \( y = \frac{1}{x} \), as \( x \to \infty \), \( y \to 0 \) and as \( x \to -\infty \), \( y \to 0 \); that is, as \( x \) approaches a large positive value or a large negative value, \( y \) approaches zero. This means that the graph of \( y = \frac{1}{x} \) will approach a \( y \)-value of zero, but never reach it, at the left and right ends of the graph. We call a line that a graph approaches but never reaches an asymptote. For the graph of \( y = \frac{1}{x} \), the line \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote.

Also, \( x \) cannot equal zero, but as \( x \) approaches a very small positive value near zero, \( y \to \infty \), and as \( x \) approaches a very small negative value, \( y \to -\infty \).

For the graph of \( y = \frac{1}{x} \), the line \( x = 0 \) (the \( y \)-axis) is a vertical asymptote.

A sketch of a hyperbola should include asymptotes, labelled with their equations. The graph of \( y = x^{-1} = \frac{1}{x} \) is shown opposite.

The equation \( y = x^{-2} \) is the same as \( y = \frac{1}{x^2} \) and its graph is called a truncus. As with the hyperbola, the domain of \( y = x^{-2} \) is \( R \setminus \{0\} \). The range is \( R^+ \) or \( (0, \infty) \), since squaring \( x \) means that \( y \) is never negative. The line \( y = 0 \) (the \( x \)-axis) is a horizontal asymptote and the line \( x = 0 \) (the \( y \)-axis) is a vertical asymptote.

The major difference between the hyperbola and the truncus is that, because \( x^2 \) is positive, \( y \) is positive. The graph of \( y = x^{-2} = \frac{1}{x^2} \) is shown opposite.
Example 1

A function has rule \( f(x) = \frac{3}{x-2} - 1 \). State the implied domain of \( f \) and hence sketch the graph of the function. State the range of the function.

Solution

The domain is \( \mathbb{R} \setminus \{2\} \).
The range is \( \mathbb{R} \setminus \{-1\} \).

Example 2

Find the largest possible set \( D \) if \( f: D \rightarrow \mathbb{R} \), where \( f(x) = \frac{1}{\sqrt{x^2 - 1}} \).

Solution

To determine what values \( x \) can take, consider the restrictions on \( x \) and therefore the values which \( x \) cannot take.
First, as \( x^2 - 1 \) is inside a square root, it must be greater than or equal to zero i.e. \( x^2 - 1 \geq 0 \).
Second, as \( \sqrt{x^2 - 1} \) is in the denominator, it cannot be equal to zero, so \( x^2 - 1 \neq 0 \).
Combining these, it follows that \( x^2 - 1 > 0 \).
Consider the graph of \( y = x^2 - 1 \). The graph is an upright parabola with \( x \)-axis intercepts at \( x = -1 \) and \( x = 1 \). So \( x^2 - 1 > 0 \) provided \( x < -1 \) or \( x > 1 \).
So \( D = \{x: x < -1\} \cup \{x: x > 1\} \) or \( D = (-\infty, -1) \cup (1, \infty) \).

exercise 2.1

1 Sketch the graphs of
   a \( y = (x - 1)^3 + 3 \)       b \( y = (x + 2)^3 - 1 \)       c \( y = 2 - (x + 1)^3 \)
2 Sketch the graphs of
   a \( y = (x + 2)^4 + 1 \)       b \( y = (x - 1)^4 - 2 \)       c \( y = -(x - 2)^4 - 3 \)
3 Sketch the graphs of
   a \( y = \frac{4}{x+3} - 2 \)       b \( y = \frac{2}{x-4} + 1 \)       c \( y = 3 - \frac{2}{x-1} \)
4 Sketch the graphs of
   a \( y = \frac{2}{(x+1)^2} - 3 \)       b \( y = \frac{3}{(x-2)^2} + 1 \)       c \( y = 2 - \frac{4}{(x-1)^2} \)
5 Sketch the graphs of
   a  \( y = \sqrt[3]{x+3} + 2 \)  
   b  \( y = \sqrt{x+1} - 4 \)  
   c  \( y = 3 - \sqrt{x-1} \)

6 Sketch the graphs of
   a  \( y = x^{1/3} \)  
   b  \( y = x^{2/3} \)  
   c  \( y = x^{1/2} \)  
   d  \( y = x^{-2/3} \)

7 A function has rule \( f(x) = \frac{2}{x+3} - 1 \). State the implied domain of \( f \) and hence sketch the graph of the function. State the range of the function.

8 Find the largest possible set \( D \) if \( f: D \to R \), where \( f(x) = \frac{\sqrt{x+2}}{x+1} \).

9 State the implied domain of \( f \) and hence sketch its graph if the function has rule
   a  \( f(x) = \sqrt{x+2} - 3 \)  
   b  \( f(x) = \frac{4}{x-3} - 2 \)  
   c  \( f(x) = \frac{3}{(x+1)^2} + 2 \)

   State the range in each case.

10 Find the largest possible set \( D \) if \( f: D \to R \), where
   a  \( f(x) = \frac{1}{\sqrt{1-x^2}} \)  
   b  \( f(x) = \sqrt{2x+1} \)  
   c  \( f(x) = \frac{\sqrt{x}}{x-4} \)

**Polynomial functions in factorised form and their graphs**

Let \( f: R \to R \), \( f(x) = p(x) \), where \( p(x) \) is a polynomial. When \( p(x) \) is in factorised form so that each factor is linear (in the form \( ax + b \)), it is very easy to read off the \( x \)-intercepts by simply finding the \( x \) values for which the expressions inside the brackets equal zero. We then use the \( x \)- and \( y \)-intercepts to sketch the graph of \( f \).

**Quadratic functions**

If a quadratic expression can be factorised, we can use this to find the \( x \)-intercept(s). If the rule can be written in the form \( y = a(x + b)^2 \), there is only one \( x \)-intercept \((-b, 0)\) and that point is also the turning point. When a parabola has two \( x \)-intercepts, its axis of symmetry is the vertical line passing through the midpoint of the line segment joining them. The turning point lies on the axis of symmetry, so the \( x \)-coordinate of the midpoint is the \( x \)-coordinate of the turning point.

**Example 1**

Sketch the graph of \( y = (x + 3)^2 \).

**Solution**

There is only one \( x \)-intercept \((-3, 0)\), so that point is also the turning point.

Substituting \( x = 0 \) gives the \( y \)-intercept \((0, 9)\).

Now sketch the graph.
Sometimes we are required to sketch the graph of a parabola and its rule is not in factorised form. One approach we can use is to first factorise the rule and then proceed as above. Another approach is to complete the square.

Example 2

Sketch the graphs of:

a. \( y = x^2 - 4x - 12 \)

b. \( y = x^2 - 4x + 12 \)

c. \( y = -x^2 + 4x + 12 \)

Solution

a. Factorising gives \( y = x^2 - 4x - 12 = (x - 6)(x + 2) \). There are two x-intercepts: \((-2, 0)\) and \((6, 0)\). The turning point has an x-coordinate halfway between \(-2\) and \(6\), which is \(2\). Substituting \(x = 2\) gives \(y = (-4)(4) = -16\), so the turning point has coordinates \((2, -16)\). Substituting \(x = 0\) gives the y-intercept \((0, -12)\). Now sketch the graph.

b. \( x^2 - 4x + 12 \) cannot be factorised so we complete the square to find the turning point.

\[
x^2 - 4x + 12 = (x^2 - 4x + 4) + 8 = (x - 2)^2 + 8
\]

So the turning point is at \((2, 8)\). Substituting \(x = 0\) gives the y-intercept \((0, 12)\). There are no x-intercepts as \(y = 0\) has no solutions. Now sketch the graph.

tip

Part a can also be done by completing the square to write the equation in turning point form, but it is easier to use the x-intercepts.

c. Factorising gives \( y = -x^2 + 4x + 12 = -(x^2 - 4x - 12) = -(x - 6)(x + 2) \). Note that this graph has the same x-intercepts as the graph in part a but will be inverted, i.e. reflected in the x-axis. Alternatively complete the square.

\[
y = -x^2 + 4x + 12
= -(x^2 - 4x - 12)
= -[(x^2 - 4x + 4) - 4 - 12]
= -[(x - 2)^2 - 16]
= -(x - 2)^2 + 16
\]

The advantage of this method is that it is easy to see the transformations that have been performed on the basic graph of \( y = x^2 \) to form the new graph. It has been reflected in the \( x \)-axis, then translated 2 units to the right and 16 units up, giving us the turning point.

**Example 3**

The parabola shown has an equation of the form \( y = ax^2 + bx + c \). Use the \( x \)- and \( y \)-intercepts to find the equation of the parabola.

**Solution**

The \( x \)-intercepts are \((-0.5, 0)\) and \((2, 0)\), so the equation must be of the form \( y = k(2x + 1)(x - 2) \). Use the \( y \)-intercept \((0, -2)\) to find the value of \( k \), giving \( k = 1 \).

So the equation is \( y = (2x + 1)(x - 2) = 2x^2 - 3x - 2 \) (multiplying out the brackets and simplifying).

**Example 4**

Sketch the graph of \( y = x^2 - 2x - 4 \), labelling the \( x \)-intercepts with their exact values.

**Solution**

Solving \( x^2 - 2x - 4 = 0 \), we use the quadratic formula to give

\[
x = \frac{2 \pm \sqrt{4 + 16}}{2}
\]

\[
= \frac{2 \pm 2\sqrt{5}}{2}
\]

\[
= (1 \pm \sqrt{5})
\]

If only approximate values of the \( x \)-intercepts are required, the PolySmlt APP on the Ti-83/84 may be helpful.
Alternatively, complete the square to get \( x^2 - 2x - 4 = (x - 1)^2 - 5 \). The \( x \)-intercepts can then be found either by factorising using difference of squares or by solving \( (x - 1)^2 = 5 \). The turning point is \((1, -5)\) as the basic graph of \( y = x^2 \) has been translated 1 unit to the right and 5 units down. Substituting \( x = 0 \) gives the \( y \)-intercept \((0, -4)\). Now sketch the graph.

**exercise 2.1 (continued)**

11 Sketch the graph of:
   a. \( y = (x + 2)^2 \)
   b. \( y = (x - 1)^2 \)
   c. \( y = -(x - 3)^2 \)
   d. \( y = -(x + 4)^2 \)
   e. \( y = (x + 1)^2 - 3 \)
   f. \( y = -(x - 2)^2 + 1 \)

12 Sketch the graph of:
   a. \( y = (x - 2)(x + 2) \)
   b. \( y = -(x - 1)(x - 2) \)
   c. \( y = (x - 2)(x - 4) \)
   d. \( y = (2x + 1)(x + 3) \)
   e. \( y = -(2x + 3)(x - 2) \)

13 By first factorising the quadratic expression, sketch the graph of:
   a. \( y = x^2 - x - 2 \)
   b. \( y = x^2 + 2x - 8 \)
   c. \( y = -x^2 + 6x - 8 \)
   d. \( y = 2x^2 - 3x - 2 \)
   e. \( y = -2x^2 + x + 6 \)

14 Sketch the graph of:
   a. \( y = -x^2 - 3x - 2 \)
   b. \( y = x^2 + 2x + 4 \)
   c. \( y = -x^2 + 6x - 10 \)
   d. \( y = 2x^2 + 5x - 3 \)

15 Each of the parabolas below has an equation of the form \( y = ax^2 + bx + c \). Use the \( x \)- and \( y \)-intercepts to find the equation of the parabola.
16 For each of the equations below, sketch the graph, labelling the \(x\)-intercepts with their exact values.
   \(a\) \(y = x^2 - 5\) \(b\) \(y = x^2 + 4x + 1\) \(c\) \(y = x^2 - 6x + 2\)

17 A parabola has its turning point at \((-2, -9)\) and its \(y\)-intercept at \((0, -5)\).
   \(a\) Find the equation of the parabola in the form \(y = a(x + b)^2 + c\).
   \(b\) Find the \(x\)-intercepts of the parabola.

18 A parabola has its turning point at \((1, -8)\) and its \(y\)-intercept at \((0, -6)\).
   \(a\) Find the equation of the parabola in the form \(y = a(x + b)^2 + c\).
   \(b\) Find the \(x\)-intercepts of the parabola.

Cubic functions

If a cubic expression is in factorised form, we can use it to find the \(x\)-intercepts. If it can be factorised there can be one, two, or three \(x\)-intercepts.

If a rule can be written in the form \(y = a(x + b)^3\), the graph will have one \(x\)-intercept and this will be a stationary point of inflection. In many other cases, for example when the cubic expression has three linear factors, the graph of the cubic will have one of the shapes shown below, depending on whether the coefficient \(a\) of \(x^3\) is positive or negative.

From the factorised form of a cubic, we can plot the \(x\)-intercepts and the \(y\)-intercept and then sketch the graph.

If a cubic function has a repeated linear factor and one other factor, then its graph will have just two \(x\)-intercepts. One of the \(x\)-intercepts must be a turning point, corresponding to the repeated factor.

**Example 5**

Sketch the graph of \(y = (x - 4)(x - 2)(x + 2)\).

**Solution**

From the factorised form, the \(x\)-intercepts are \((-2, 0), (2, 0), (4, 0)\). Substituting \(x = 0\) gives the \(y\)-intercept \((0, 16)\). Now sketch the graph.
**Example 6**

Sketch the curve with equation \( y = (x - 2)^2(x + 3) \).

**Solution**

The \( x \)-intercepts are (2, 0) and (-3, 0), and (2, 0) is a turning point as \((x - 2)^2\) is a repeated linear factor. Substituting \( x = 0 \) gives the \( y \)-intercept (0, 12). Now sketch the graph.

Sometimes we are required to sketch a cubic graph and the cubic expression is not in factorised form. One approach is to first factorise the equation and then proceed as before.

**Example 7**

Sketch the graph of \( y = x^3 + 4x^2 + x - 6 \). Give coordinates of the turning points correct to 2 decimal places.

**Solution**

Using the factor theorem, we find that \( y(1) = 0 \) so \((x - 1)\) is a factor. Dividing \((x - 1)\) into \( x^3 + 4x^2 + x - 6 \) gives \( x^2 + 5x + 6 \), so

\[
\begin{align*}
y &= x^3 + 4x^2 + x - 6 \\
&= (x - 1)(x^2 + 5x + 6) \\
&= (x - 1)(x + 2)(x + 3).
\end{align*}
\]

There are three \( x \)-intercepts: (-3, 0), (-2, 0) and (1, 0). Substituting \( x = 0 \) gives the \( y \)-intercept (0, -6). Now sketch the graph.

To find the coordinates of the turning points, use the maximum (or minimum) command of your graphics calculator. The following screenshots show that the required coordinates, correct to 2 decimal places, are (-2.54, 0.88) and (-0.13, -6.06).
Example 8

The cubic graph at right has an equation of the form \( y = ax^3 + bx^2 + cx + d \). Use the \( x \)- and \( y \)-intercepts to find its equation.

Solution

The \( x \)-intercepts are \((-3, 0), (1, 0) \) and \((3, 0)\) so the equation is \( y = a(x + 3)(x - 1)(x - 3) \). We use the \( y \)-intercept \((0, 18)\) to find the value of \( a \). Substituting we get
\[
18 = a(3)(-1)(-3) \\
a = 2
\]

So the equation is \( y = 2(x - 3)(x - 1)(x + 3) \). Expanding gives \( y = 2x^3 - 2x^2 - 18x + 18 \).

Example 9

Consider the cubic graph with equation \( y = (x^2 + 1)(x - 2) \).

\( a \) Find the axes intercepts.

\( b \) Use technology to help sketch the graph.

Solution

\( a \) As \((x^2 + 1)\) has no real factors, \( x^2 + 1 \neq 0 \) for any real value of \( x \), so the only \( x \)-intercept is \((2, 0)\). The \( y \)-intercept is \((0, -2)\). We can learn more about the shape of the graph by considering that as \( x \to \infty \), \( y \to \infty \); and as \( x \to -\infty \), \( y \to -\infty \).

\( b \) As there is just one \( x \)-intercept, it is not possible to determine whether the graph will have two turning points or a single horizontal point of inflection, or neither, without more sophisticated methods such as calculus (see chapter 5). We use a graphics calculator to produce sufficient detail. The graph to the right is the result. It has two turning points.

exercise 2.1

19 Sketch the graph of:

\( a \) \( y = (x - 1)(x + 1)(x + 3) \)  
\( b \) \( y = (x - 1)(x - 2)(x - 3) \)  
\( c \) \( y = (x - 2)(x - 3)(x + 3) \)  
\( d \) \( y = -(x + 1)(x + 3)(x + 4) \)

20 Sketch the cubic graph with equation:

\( a \) \( y = (x + 1)^2(x + 4) \)  
\( b \) \( y = (x - 3)^2(x + 1) \)  
\( c \) \( y = (x - 2)(x - 1)^2 \)  
\( d \) \( y = (1 - x)(x + 2)^2 \)

21 For each of the equations below, sketch the graph, labelling the \( x \)-intercepts with their exact values.

\( a \) \( y = (x + 1)(x^2 - 3) \)  
\( b \) \( y = x^3 + 2x^2 - 6x \)  
\( c \) \( y = x^3 - x^2 - 3x + 3 \)
22 Sketch the graph of each of the following. Give coordinates of any turning points correct to 2 decimal places.
   a  \( y = x^3 + 2x^2 - 5x - 6 \)
   b  \( y = x^3 - 2x^2 - 5x - 6 \)
   c  \( y = x^3 + 5x^2 + 7x + 3 \)
   d  \( y = x^3 - 7x^2 + 11x - 5 \)

23 Each of the cubic graphs below has an equation of the form \( y = ax^3 + bx^2 + cx + d \). Use the \( x \)- and \( y \)-intercepts to find the equation in each case.

24 Consider the cubic graph with equation:
   a  \( y = (x^2 + 4)(x + 1) \)
   b  \( y = (x^2 + 2x + 2)(x - 3) \)

In each case, find the axes intercepts and use technology to help sketch the graph.

Quartic functions

If a quartic expression is in factorised form, we can use this to find the \( x \)-intercepts. There can be zero, one, two, three or four \( x \)-intercepts. If a rule can be written in the form \( y = a(x + b)^4 \), the graph will have one \( x \)-intercept and this will be the turning point.

If a quartic expression \( f(x) \) can be factorised to give four different linear factors, the graph of \( y = f(x) \) has four \( x \)-intercepts. We plot the \( x \)-intercepts and the \( y \)-intercept and then sketch the graph.

If \( f(x) \) has four different linear factors and a positive coefficient of \( x^4 \), the graph of \( y = f(x) \) has the general shape shown here:

If \( f(x) \) has four different linear factors and a negative coefficient of \( x^4 \), the graph of \( y = f(x) \) has the general shape shown here:

Notice that there are three turning points in each case.
Example 10

Sketch the curve with equation \( y = (x + 3)(x + 1)(x - 1)(x - 2) \). Give coordinates of the turning points correct to 2 decimal places.

Solution

The \( x \)-intercepts are \((-3, 0), (-1, 0), (1, 0), (2, 0)\). Substituting \( x = 0 \) gives the \( y \)-intercept \((0, 6)\).

Now sketch the graph.

Using a graphics calculator the turning points have coordinates, correct to 2 decimal places of \((-2.25, -12.95), (-0.07, 6.04) \) and \((1.57, -2.88)\). The first of these is shown here.

If a quartic function has a once repeated linear factor, then its graph will have just three \( x \)-intercepts. The \( x \)-intercept corresponding to the repeated factor is also a turning point.

Example 11

Sketch the graph of \( y = (x - 1)^2(x + 2)(x - 3) \).

Solution

The \( x \)-intercepts are \((-2, 0), (1, 0) \) and \((3, 0) \) and \((1, 0) \) is a turning point as \((x - 1)^2 \) is a repeated linear factor. Substituting \( x = 0 \) gives the \( y \)-intercept \((0, -6) \). Now sketch the graph.

If a quartic function has two repeated linear factors, then its graph will have just two \( x \)-intercepts. The \( x \)-intercepts must be turning points, corresponding to the repeated factors.
Example 12

Sketch the graph of \( y = (x - 2)^2(x + 1)^2 \).

Solution

The \( x \)-intercepts are \((-1, 0)\) and \((2, 0)\) and both are turning points as \((x - 2)^2\) and \((x + 1)^2\) are repeated linear factors. Substituting \( x = 0 \) gives the \( y \)-intercept \((0, 4)\). Sketch the graph.

Example 13

Sketch the curve with equation \( y = (x + 2)^3(x - 1) \).

Solution

The \( x \)-intercepts are \((-2, 0)\) and \((1, 0)\) and \((-2, 0)\) is a stationary point of inflection, as \((x + 2)^3\) is a triple repeated linear factor. Substituting \( x = 0 \) gives the \( y \)-intercept \((0, -8)\). Sketch the graph.

Example 14

Consider the curve with equation \( y = (x^2 + 2)(x - 1)(x + 1) \).

a Find the axes intercepts.

b Use technology to help sketch the graph.

Solution

a As \( x^2 + 2 \neq 0 \) for any real values of \( x \), the only \( x \)-intercepts are \((-1, 0)\) and \((1, 0)\). The \( y \)-intercept is \((0, -2)\).

b As in example 9, it is not possible to determine the number of turning points. Using a graphics calculator will provide sufficient detail. The graph to the right is the result. It has just one turning point at \((0, -2)\).
Higher degree polynomial functions

When a polynomial function has a degree that is higher than 4, we proceed in the same way as before. That is, first write the polynomial expression in factorised form. For each factor which is linear (in the form $ax + b$) we can read off the corresponding $x$-intercept by simply finding the $x$ value for which the expression inside the brackets equals zero. Then use the $x$- and $y$-intercepts and the help of technology where necessary to sketch the graph.

**Example 15**

Sketch the graph of $y = x^5 - x^3$.

**Solution**

Factorise $y = x^5 - x^3$ to get $y = x^3(x^2 - 1)$ and then factorise further to get $y = x^3(x - 1)(x + 1)$. The $x$-intercepts are $(0, 0)$, $(0, -1)$ and $(1, 0)$. The $y$-intercept is $(0, 0)$. The $x^3$ factor indicates that there is a stationary point of inflection at the origin. We can learn more about the shape of the graph by considering that as $x \to -\infty$, $y \to -\infty$; and as $x \to +\infty$, $y \to +\infty$. Then use technology to help sketch the graph.

**exercise 2.1**

25 Sketch the graph of:

a $y = (x - 1)(x + 2)(x + 3)(x - 2)$

b $y = (x - 2)(x + 2)(x + 1)(x - 1)$

c $y = -(x - 3)(x + 1)(x + 3)(x - 2)$

26 Sketch the curve with equation shown below. Give coordinates of the turning points correct to 2 decimal places.

a $y = (x + 1)^2(x + 3)(x - 2)$

b $y = (x - 2)^2(x + 1)(x - 1)$

c $y = (x + 1)(1 - x)(x - 3)^2$

27 Sketch the graph of:

a $y = (x - 1)^2(x + 2)^2$

b $y = (x + 1)^2(x - 3)^2$

c $y = -(x + 3)^2(x + 1)^2$

28 Sketch the graph of:

a $y = (x - 1)^3(x + 3)$

b $y = (x + 1)^3(x + 2)$

c $y = (x + 3)(1 - x)^3$

29 For each of the equations below, sketch the graph, labelling the $x$-intercepts with their exact values.

a $y = (x - 1)(x + 3)(x^2 - 3)$

b $y = x^4 - 8x^2 + 15$

c $y = x^4 - 3x^2 + 2$

30 A quartic function has equation $y = ax^4 + bx^3 + cx^2 + dx + e$. Its graph cuts the $x$-axis at $(-1, 0)$ and $(2, 0)$. One of these intercepts is a stationary point of inflection. If the graph passes through $(1, 16)$, find $a$, $b$, $c$, $d$ and $e$.
For each of the quartic graphs below, use the x- and y-intercepts to find an equation in factorised form.

![Graph A](image1)

For each of the quartic graphs below, use the x- and y-intercepts to find its equation in the form \( y = x^4 + bx^3 + cx^2 + dx + e \).

![Graph B](image2)

Consider the curve with equation:

- \( a \quad y = (x^2 + 3)(x + 2)(x + 1) \)
- \( b \quad y = (x^2 + x + 2)(x - 1)(x - 3) \)

In each case, find the axes intercepts and use technology to help sketch the curve.

Sketch the graph with equation:

- \( a \quad y = x^5 - x^4 - 2x^3 \)
- \( b \quad y = x^6 - x^2 \)
- \( c \quad y = x^5 + x^4 - 3x^3 - 3x^2 \)
Inequalities

For a function with rule \( y = f(x) \), it is often necessary to solve an inequality such as \( y < c \) or \( y \geq d \). This can always be done graphically with technology (to give at least an approximate answer), or algebraically with the help of a graph, or sometimes just with algebra. The approach will often be determined by the complexity of the function rule.

**Example 16**

Solve for \( x \):

- **a** \(-2x + 1 \geq 5\)
- **b** \(x^2 + 3x - 4 < 0\)

**Solution**

- **a** This can be readily solved with algebra.
  
  \[-2x + 1 \geq 5\]
  
  \[-2x \geq 4\] \((subtracting 1 from both sides)\)
  
  \[x \leq -2\] \((dividing both sides by \(-2)\)

- **b** \(x^2 + 3x - 4 < 0\)
  
  \((x + 4)(x - 1) < 0\)

  Now \((x + 4)(x - 1) = 0\) if \(x = -4\) or \(x = 1\).

  If \(x < -4\), both bracketed terms are negative. Since the product of two negative numbers is positive, it follows that \((x + 4)(x - 1) > 0\) if \(x < -4\).

  Similarly if \(x > 1\), both bracketed terms are positive, and so their product is positive.

  If \(-4 < x < 1\), the first bracketed term is positive and the second is negative, so the product is negative.

  Hence \(x^2 + 3x - 4 < 0\) if \(-4 < x < 1\).

**Warning** Reverse!

Recall that when multiplying or dividing both sides of an inequality by a negative number, the inequality sign is reversed.

**Tip**

The results in example 16 are easy to check graphically. In the screenshot below left, the graph of \(y = -2x + 1\) is shown together with a horizontal line at height 5. It is easy to see that \(y \geq 5\) when \(x \leq -2\). In the screenshot below right, the parabola with the equation \(y = x^2 + 3x - 4\) is shown with the first \(x\)-axis intercept at \((-4, 0)\). It is evident that \(y < 0\) if \(-4 < x < 1\).
Example 17
If \( y = x^2 - 1 \), find \( \{ x : y > 1 \} \) using:

a technology. 

b algebra.

Solution

a The screenshot at the right shows the graph \( y = x^2 - 1 \) and one of its intersections with a horizontal line at height 1, namely \( x = -1.414 \). The second intersection occurs when \( x = 1.414 \). Hence the approximate solution is \( \{ x : x < -1.414 \} \cup \{ x : x > 1.414 \} \) or \( (-\infty, -1.414) \cup (1.414, \infty) \).

b If \( y > 1 \), then:
\[
\begin{align*}
  x^2 - 1 &> 1 \\
  x^2 - 2 &> 0 \\
  (x + \sqrt{2})(x - \sqrt{2}) &> 0
\end{align*}
\]
Using the reasoning of example 15 or the graph in part a here, the solution is \( \{ x : x < -\sqrt{2} \} \cup \{ x : x > \sqrt{2} \} \) or \( (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) \).

The advantage of the algebraic method, when available, is that exact solutions can be obtained as here.

exercise 2.1

35 Solve for \( x \):

a \( -x - 1 \leq 2 \)  

b \( 3 - 3x > 3 \)  

c \( x^2 - 3x + 2 < 0 \)  

d \( x^2 - x - 6 \geq 0 \)  

e \( x^2 < 7 \)  

36 If \( y = x^2 - 2 \), find \( \{ x : y \leq 3 \} \) using:

a technology. 

b algebra.

37 If \( y = \frac{3}{x - 2} \), find the values of \( x \) for which \( y < 2 \).  
(Hint: a graph is very useful in this question.)

38 Find \( \left\{ x : \frac{2}{3x+1} > \frac{1}{2} \right\} \).
A cannon is on the edge of a cliff (at A), overlooking the sea. In the graph below, the origin is placed at the intersection of the vertical cliff face (y-axis) and the horizontal sea level (x-axis). It should be noted that x and y are measured in hundreds of metres. The point (2, 0.24) is therefore 200 m from the cliff and 24 m above sea level.

A ship of invaders is positioned at the point (3, 0). To hit the ship, the cannon ball must travel along the path shown below which is part of the graph of a cubic function. If the entire graph was shown, there would be a turning point at (−2, 0).

a Show that the equation of the path can be given as \( y = f(x) = k(x^3 + x^2 - 8x - 12) \), where \( k \) is a constant.

b Assuming that the cannon ball hits the enemy ship and then stops, write down the domain of \( f \).

c Use the fact that the cannon ball passes through the point B(2, 0.24) to find \( k \).

d Hence find the height, in metres, from which the cannon ball is fired.

e Find the maximum height the cannon ball reaches to the nearest tenth of a metre.

Another enemy ship is positioned at the point (6, 0). This ship is equipped with a weapon that can fire a missile at great speed so that the path of the missile can be modelled by a straight line.

f Find the equation of the straight line for the missile to collide with the cannon ball at the point B if the timing is right.

g If the missile was fired along the straight line found in part f, but did not hit the cannon ball, how far (in metres) above the cannon would it pass?
Exponential and logarithmic functions

Exponential functions

An exponential function is defined as having a rule \( y = a^x, \ a \in R^+ \setminus \{1\} \). That is, \( a \) can take the value of any positive real number other than 1 (as \( a = 1 \) would give a straight line).

If \( a > 1 \), as \( x \to -\infty \), \( y \to 0 \) and, as \( x \to \infty \), \( y \to \infty \) and there are no turning points; so the function is increasing. The graphs of \( y = 2^x \) and \( y = 10^x \) are shown below:

If \( a < 1 \), as \( x \to -\infty \), \( y \to \infty \) and as \( x \to \infty \), \( y \to 0 \) and there are no turning points; so the function is decreasing. The graph of \( y = \left(\frac{1}{2}\right)^x \) is shown below:

\[
\begin{array}{c}
\text{tip} \\
y = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x} \text{ so the equation } y = \left(\frac{1}{2}\right)^x \text{ can be formed from the equation of } y = 2^x \text{ by replacing } x \text{ with } -x. \text{ So the graph of } y = \left(\frac{1}{2}\right)^x \text{ is the reflection of the graph of } y = 2^x \text{ in the } y\text{-axis and vice versa.} \\
\text{Also, } y = \left(\frac{1}{2}\right)^x \text{ is the reciprocal function of } y = 2^x \text{ so that when the value of one function gets larger, the other gets smaller, and vice versa.}
\end{array}
\]
An exponential function with rule $y = a^x$ has the following properties:

- Its graph passes through $(0, 1)$ as $a^0 = 1$.
- Its graph passes through $(1, a)$ as $a^1 = a$.
- Its graph has a horizontal asymptote at $y = 0$.
- The domain of the function is $\mathbb{R}$.
- The range of the function is $\mathbb{R}^+$.

### Exponential functions with base $e$

We learnt in chapter 1 that $e$ is a constant that has an approximate value of 2.718282. The number $e$ is of great interest because it has several important mathematical features, some of which we will learn about in this course.

The graph of $y = e^x$ has the same properties as those shown above for exponential functions. It is shown here with the graph of $y = 2^x$.

#### Example 1

Sketch the graph of $y = e^{2x}$.

**Solution**

$y = e^{2x} = (e^2)^x$ and so is of the form $y = a^x$.

The graph must pass through $(0, 1)$ and $(1, e^2)$.

#### Example 2

Sketch the curve with equation:

<table>
<thead>
<tr>
<th></th>
<th>$a$ $y = e^x + 1$</th>
<th>$b$ $y = e^{2-x}$</th>
</tr>
</thead>
</table>

**Solution**

- **a** Compared with the graph of $y = e^x$, 1 has been added to each $y$ value so the graph is translated up by 1 unit.
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\[ b \ y = e^{2-x} = e^2 \times e^{-x} \]. Compared with the graph of \( y = e^{-x} \), the \( y \) values have been multiplied by \( e^2 \) so the graph of \( y = e^{-x} \) has been stretched vertically (dilated) to form the graph of \( y = e^{2-x} \). To find the \( y \)-intercept, we substitute \( x = 0 \) and find that \( y = e^2 \) so the \( y \)-intercept is \((0, e^2)\). (Alternatively, note that the graph of \( y = e^{-x} \) has been translated 2 units to the right.)

**exercise 2.2**

1. Sketch the graph of:
   a. \( y = 3^x \)
   b. \( y = 5^x \)
   c. \( y = 7^x \)

2. Sketch the graph of:
   a. \( y = 3 \times 2^x \)
   b. \( y = 2 \times 3^x \)
   c. \( y = 2 \times 5^x \)

3. Sketch the curve with equation:
   a. \( y = 3^{-x} \)
   b. \( y = 5^{-x} \)
   c. \( y = 7^{-x} \)

4. Sketch the graph of:
   a. \( y = 3 \times 2^{-x} \)
   b. \( y = 2 \times 3^{-x} \)
   c. \( y = 2 \times 5^{-x} \)

5. Sketch the graph of:
   a. \( y = e^{3x} \)
   b. \( y = e^{4x} \)
   c. \( y = e^{1.5x} \)

6. Sketch the curve with equation:
   a. \( y = e^{x} - 3 \)
   b. \( y = e^{x} + 2 \)
   c. \( y = -e^{x} - 1 \)

7. Sketch the curve with equation below, labelling the \( y \)-intercept with its exact value in each case.
   a. \( y = e^{x+1} \)
   b. \( y = e^{x-2} \)
   c. \( y = e^{1-x} \)

8. Sketch the graph of:
   a. \( y = e^{x/3} \)
   b. \( y = \sqrt{e^{x}} \)
   c. \( y = e^{x/4} \)

**Example 3**

The graph of \( y = Ae^{kt} \) has its \( y \)-intercept at \((0, 4)\) and passes through \((3, 2)\). Find the exact values of \( A \) and \( k \).

**Solution**

When \( t = 0 \), \( y = 4 \), so \( 4 = Ae^0 \) and \( A = 4 \). So \( y = 4e^{kt} \).

Substituting the coordinates of the point \((3, 2)\) into the equation:

\[
\begin{align*}
2 &= 4e^{3k} \\
e^{-k} &= 0.5 \\
3k &= \log e 0.5 \\
k &= \frac{1}{3} \log e 0.5
\end{align*}
\]
Example 4

A radioactive substance is decaying in a laboratory. Initially 5 kg of the radioactive substance was present and 4 days later half of it remained. The mass \( R \) kg of radioactive substance at time \( t \) days after it was placed in the laboratory can be modelled by \( R = R_0 e^{-kt} \), \( t \geq 0 \).

a. State the value of \( R_0 \).

b. Find the exact value of \( k \).

c. Sketch the graph of \( R \).

d. To the nearest hour, how long after it was placed in the laboratory had 80% of the substance decayed?

Solution

a. \( R(0) = R_0 e^0 = 5 \) so \( R_0 = 5 \).

b. When \( t = 4 \), \( R = 2.5 \). Substituting gives:

\[
\begin{align*}
5 e^{-4k} &= 2.5 \\
e^{-4k} &= 0.5 \\
-4k &= \log_e 0.5 \\
k &= \frac{1}{4} \log_e 0.5 \\
&= \frac{1}{4} \log_e 0.5^{-1} \\
&= \frac{1}{4} \log_e 2
\end{align*}
\]

c. When sketching the graph, we must label the axes \( R \) and \( t \) and only sketch the graph for \( t \geq 0 \), since the model is only specified on this domain.

d. If 80% of the substance has decayed, 20% remains, so \( R = 1 \). Plot the graph of \( R = 1 \) on the same set of axes as \( R \) and find the point of intersection.

At the point of intersection \( t = 9.288 \).
Now 9.288 days is the same as 9 days and 0.288 \times 24 \text{ hours} = 9 \text{ days} and 7 \text{ hours}.
So 80% of the substance had decayed after 9 days and 7 hours.
9 The graph of \( y = Ae^{kt} \) has its \( y \)-intercept at \((0, 2)\) and passes through \((4, 7)\). Find the exact values of \( A \) and \( k \).

10 The graph of \( y = Ae^{-kt} \) has its \( y \)-intercept at \((0, 7)\) and passes through \((5, 3)\). Find the exact values of \( A \) and \( k \).

11 A spoonful of sugar is dissolving in hot water. The initial mass of the sugar was 7 grams and 5 minutes later the mass present had halved. The mass of sugar, \( S \) grams, at time \( t \) minutes after it was placed in the hot water can be modelled by \( S(t) = S_0e^{-kt}, \, t \geq 0 \).

   a State the value of \( S_0 \).
   b Find the exact value of \( k \).
   c Sketch the graph of \( S(t) \).
   d To the nearest second, how long after the sugar was placed in the hot water was its mass 1 gram?

12 A crystal is growing in a beaker. A 2 mg crystal was placed in the beaker and 8 hours later the mass present had doubled. The mass of crystal, \( C \) mg, in the beaker at time \( t \) hours after the crystal was placed in the beaker can be modelled by \( C(t) = C_0e^{kt}, \, t \geq 0 \).

   a State the value of \( C_0 \).
   b Find the exact value of \( k \).
   c Sketch the graph of \( C(t) \).
   d To the nearest minute, how long after the crystal was placed in the beaker was 5 mg of crystal present?

13 The volume of chlorine, \( C \) litres, in a swimming pool at time \( t \) hours after it was placed in the pool can be modelled by \( C(t) = C_0e^{kt}, \, t \geq 0 \). The volume of chlorine in the pool is decreasing. Initially the volume of chlorine in the pool was 3 litres, 8 hours later the volume was 2.5 litres.

   a State the value of \( C_0 \).
   b Find the exact value of \( k \).
   c Exactly how many litres of chlorine were present in the pool 16 hours after the chlorine was added to the pool?
   d Sketch the graph of \( C(t) \).
   e To the nearest minute, how long after the chlorine was added to the pool was there 2 litres remaining?

Logarithmic functions

A logarithmic function is defined as having a rule \( y = \log_a x, \, a \in R^+ \setminus \{1\} \). That is, \( a \) can take the value of any positive real number other than 1 (as \( a = 1 \) would give a straight line).

We recall that we can only take the logarithm of a positive number so the domain of a logarithmic function is \( R^+ \). We can confirm this as follows: \( y = \log_a x \) is equivalent to \( x = a^y \), so if \( a > 0 \), \( a^y > 0 \) for all \( y \) values, so \( x > 0 \).
As \( x \to 0 \), \( y \to -\infty \) and as \( x \to \infty \), \( y \to \infty \) and there are no turning points; so the function is increasing. The graphs of \( y = \log_2 x \) and \( y = \log_{10} x \) are shown:

The shape of the graph of \( y = \log_2 x \) can be seen to be similar in some way to that of the graph of \( y = 2^x \). The graph of \( y = \log_2 x \) is actually the reflection of the graph of \( y = 2^x \) in the line \( y = x \).

A logarithmic function with rule \( y = \log_a x \) has the following properties.

- Its graph passes through \((1, 0)\) as \( \log_a 1 = 0 \) (as \( a^0 = 1 \)).
- Its graph passes through \((a, 1)\) as \( \log_a a = 1 \) (as \( a^1 = a \)).
- Its graph has a vertical asymptote at \( x = 0 \) (\( \log_a x \) is only defined for \( x > 0 \)).
- The domain of the function is \( \mathbb{R}^+ \).
- The range of the function is \( \mathbb{R} \).

**Logarithmic functions with base \( e \)**

Logarithms to base \( e \) are written as \( \log_e x \) or sometimes as \( \ln x \), which is short for log natural of \( x \).

The graph of \( y = \log_e x \) has the same properties as those shown above for logarithmic functions. It is shown here with the graph of \( y = \log_2 x \).

**tip**

To plot the graph of \( y = \log_{10} x \) with a graphics calculator, use the **LOG** key. To plot the graph of \( y = \log_a x \), use the **LN** key. To plot the graph of \( y = \log_2 x \), use the change of base rule. For example, using base \( 10 \), recall that:

\[
\log_2 x = \frac{\log_{10} x}{\log_{10} 2}
\]

The graphs of \( y = \log_2 x \) and \( y = \log_{10} x \) are shown in the calculator screen below left. The graphs of \( y = \log_2 x \) and \( y = \log_e x \) are shown in the calculator screen below right.
Example 5

Sketch the graph of \( y = \log_e 2x \).

Solution

Since \( \log_e 1 = 0 \), the \( x \)-intercept occurs when \( 2x = 1 \), i.e. when \( x = \frac{1}{2} \). Also, \( \log_e e = 1 \), so \( y = 1 \) when \( 2x = e \), i.e. when \( x = \frac{e}{2} \). As we can only take the log of a positive number \( 2x > 0 \) and so \( x > 0 \). Thus there is a vertical asymptote at \( x = 0 \). With this knowledge the graph can be sketched.

Example 6

Sketch the curve with equation \( y = \log_e (x + 1) \).

Solution

Since \( \log_e 1 = 0 \), the \( x \)-intercept occurs when \( x + 1 = 1 \), or \( x = 0 \). Also, as \( \log_e e = 1 \), \( y = 1 \) when \( x + 1 = e \), so \( x = e - 1 \). As we can only take the log of a positive number, \( x + 1 > 0 \) and so \( x > -1 \). Thus there is a vertical asymptote at \( x = -1 \). With this knowledge, the graph can be sketched.

(The graph can also be sketched by noting that the graph of \( y = \log_e (x + 1) \) can be formed by translating the graph of \( y = \log_e x \) horizontally left by 1 unit.)

Example 7

The graph of \( y = A \log_e (x + b) \) has its \( x \)-intercept at \( (2, 0) \) and passes through \( (5, 2) \). Find the exact values of \( A \) and \( b \).

Solution

When \( x = 2 \), \( y = 0 \), so \( 0 = A \log_e (2 + b) \). Since the graph passes through \( (5, 2) \), \( A \neq 0 \). Thus:

\[
\log_e (2 + b) = 0
\]

\[
2 + b = 1
\]

\[
b = -1
\]

So \( y = A \log_e (x - 1) \).

Substituting the point \( (5, 2) \) into the equation gives \( 2 = A \log_e 4 \). So \( A = \frac{2}{\log_e 4} \).
exercise 2.2

14 Sketch the graph of:
   a \( y = \log_e 3x \)  
   b \( y = \log_e 4x \)  
   c \( y = \log_e 5x \)

15 Sketch the curve with equation below, and state the implied domain in each case.
   a \( y = \log_e (x + 2) \)  
   b \( y = \log_e (x - 1) \)  
   c \( y = \log_e (x - 3) \)

16 Sketch the curve with equation below, and state the implied domain in each case.
   a \( y = \log_e (2x + 1) \)  
   b \( y = \log_e (2x - 1) \)  
   c \( y = \log_e (3x - 2) \)

17 The graph of \( y = A \log_e (x + b) \) has its \( x \)-intercept at \((-1, 0)\) and passes through \((3, 4)\).
   Find the exact values of \( A \) and \( b \).

18 The graph of \( y = A \log_e (x + b) \) has its \( x \)-intercept at \((3, 0)\) and passes through \((6, 4)\).
   Find the exact values of \( A \) and \( b \).

19 Find the largest possible set \( D \) if \( f: D \to R \), where:
   a \( f(x) = \log_e (1 - 4x) \)  
   b \( f(x) = \log_e (3x - 7) \)  
   c \( f(x) = \log_e (x^2 - 4) \)

20 The number of parts per million, \( x \), of \( E. coli \) in a stream \( t \) hours after a pollutant containing \( E. coli \) is introduced is modelled by \( x(t) = \log_e (t + e^2) \), \( t \geq 0 \).
   a How many parts per million of \( E. coli \) are introduced into the stream?
   b To the nearest tenth of a part per million, how many parts per million of \( E. coli \) are in the stream when \( t = 10 \)?
   c Sketch the graph of \( x \) versus \( t \).
   d To the nearest hour, how long after the \( E. coli \) is introduced to the stream are there 4 parts per million of \( E. coli \)?

analysis task 2—
the bearded anaconda

A type of snake called the bearded anaconda is found in parts of Africa. Part of its habitat has been destroyed and as a result its numbers are decreasing. A detailed study on 1 January 1990 showed that there were 350 bearded anacondas in the wild. Five years later 320 bearded anacondas remained. The population function, \( x \), of bearded anacondas \( t \) years after 1 January 1990 is modelled by the rule \( x(t) = x_0 e^{-kt} \).

a If the model is assumed to be valid until 1 January 2050, what is the domain of the function?

b State the value of \( x_0 \).

c Show that the exact value of \( k \) is \( k = \frac{1}{5} \log_e \frac{35}{32} \).

d Sketch the graph of \( x(t) \).

e What does the model predict the population to be on 1 January 2020?

f During what year does the model predict the population will drop below 150?
2.3

**Circular functions**

**Sine and cosine graphs**

The **sine function** has rule \( y = \sin x \) and domain \( R \).

The **cosine function** has rule \( y = \cos x \) and domain \( R \).

The graph of \( y = \sin x \), for \(-2\pi \leq x \leq 2\pi\), is shown below.

![Graph of \( y = \sin x \)](image)

The graph of \( y = \cos x \), for \(-2\pi \leq x \leq 2\pi\), is shown below.

![Graph of \( y = \cos x \)](image)

Although each graph is drawn for \(-2\pi \leq x \leq 2\pi\), each continues in a similar way over their domain \( R \): that is, their form for \(-2\pi\) to 0 is the same as for 0 to \(2\pi\) and for \(2\pi\) to \(4\pi\), and so on. As \( x \) increases the shape of each curve repeats over an interval of \(2\pi\). So we say that sine and cosine are **periodic functions** with **period** \(2\pi\).

Each graph lies between \(-1\) and 1; we say that each of the graphs has **amplitude** 1, the distance from their mean position, which in this case is the \(x\)-axis. It follows that the range of each function is \([-1, 1]\).

Note that each graph has the same shape; one is a horizontal translation of the other.

**Example 1**

Sketch the curve with equation \( y = 3 \sin x \), \(0 \leq x \leq 2\pi\).

**Solution**

Compared with the graph of \( y = \sin x \) the \( y \) values have been multiplied by 3, so the range is \([-3, 3]\), and the amplitude is equal to 3. This is the same as saying that there has been a vertical dilation of 3.
Example 2
Sketch the graph of $y = \cos 2x$, $0 \leq x \leq 2\pi$.

Solution
Find the $x$-intercepts by solving $\cos 2x = 0$, $x \in [0, 2\pi]$.

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

The $y$-intercept is 1.

Use these values to sketch the graph.
(We could also sketch the graph by noting that there has been a horizontal dilation of $\frac{1}{2}$ of the cosine curve.)

Note that the period of this graph is $\pi$.

Example 3
Sketch the curve with equation $y = 4 \sin \frac{x}{2}$, $0 \leq x \leq 2\pi$.

Solution
The amplitude is 4. $P = \frac{2\pi}{\frac{1}{2}} = 2\pi \times 2 = 4\pi$. Alternatively, the angle is being ‘fed in’
to the sine function at half the rate as for $y = \sin x$, so the graph repeats in twice the distance, i.e. $P = 4\pi$.
(We could also sketch the graph by noting that there has been a vertical dilation of 4 and a horizontal dilation of 2.)
Example 4

Sketch the graph of $y = 3 \cos x + 2$, $0 \leq x \leq 2\pi$.

Solution

The amplitude is 3 and $P = 2\pi$. Compared with the graph of $y = 3 \cos x$, 2 has been added to each $y$ value, which is equivalent to translating that graph vertically by 2 units. The range of the function is therefore $[-3 + 2, 3 + 2] = [-1, 5]$.

(We could also sketch the graph by considering that there has been a vertical dilation of 3 and the graph has been translated up 2.)

Warning: Order is important!

$3 \cos x + 2$ means start with $x$, find its cosine, multiply by 3 and finally add 2. Compare this with $3 \cos (x + 2)$, where 2 is added to $x$ before finding the cosine.

Example 5

Sketch the graph of $y = -2 \cos x - 1$, $0 \leq x \leq 2\pi$.

Solution

The amplitude is 2 and $P = 2\pi$. We have reflected the graph of $y = \cos x$ in the $x$-axis and subtracted 1 from each of the $y$ values, which is equivalent to translating the graph vertically by $-1$ units. The range of the function is therefore $[-2 -1, 2 -1] = [-3, 1]$.

Exercise 2.3

1 Sketch the graph of:
   a $y = 2 \sin x$, $0 \leq x \leq 2\pi$
   b $y = 3 \cos x$, $0 \leq x \leq 2\pi$
   c $y = 4 \cos x$, $-\pi \leq x \leq \pi$
   d $y = 5 \sin x$, $-\pi \leq x \leq \pi$

2 Sketch the graphs of:
   a $y = \sin 2x$, $0 \leq x \leq 2\pi$
   b $y = \cos 3x$, $0 \leq x \leq 2\pi$
   c $y = \sin 4x$, $-\pi \leq x \leq \pi$
   d $y = \cos \frac{x}{2}$, $-2\pi \leq x \leq 2\pi$

3 Sketch the graphs with equations:
   a $y = 3 \sin 2x$, $0 \leq x \leq 2\pi$
   b $y = 2 \cos 3x$, $0 \leq x \leq 2\pi$
   c $y = 0.5 \sin 4x$, $-\pi \leq x \leq \pi$
   d $y = 1.4 \cos \frac{x}{2}$, $-2\pi \leq x \leq 2\pi$
4 Sketch the graphs of:
   a \( y = 2 \sin x + 3, \ 0 \leq x \leq 2\pi \)
   b \( y = 4 \cos x - 2, \ 0 \leq x \leq 2\pi \)
   c \( y = 3 \cos x + 4, \ -\pi \leq x \leq \pi \)
   d \( y = 3 \sin x - 3, \ -\pi \leq x \leq \pi \)

5 Sketch the graphs of:
   a \( y = -2 \sin x + 3, \ 0 \leq x \leq 2\pi \)
   b \( y = -4 \cos x - 2, \ 0 \leq x \leq 2\pi \)
   c \( y = -3 \sin x + 4, \ -\pi \leq x \leq \pi \)

6 Each graph below has an equation in the form \( y = a \sin bx + c \), \( x \in [0, 2\pi] \). Find the values of \( a \), \( b \) and \( c \) where \( a \) and \( c \) are integers, in each case.

   a
   
   b
   
   c

7 The equations of a set of trigonometric graphs (graphs of circular functions) are known to be in the form \( y = a \cos bx + c \). Find the values of \( a \) (\( a > 0 \)), \( b \) and \( c \) if the range and period respectively are as given.
   a \([-3, 9] \), \( \pi \)
   b \([-5, 1] \), \( 4\pi \)
   c \([1, 6] \), \( \frac{\pi}{3} \)
   d \([-7, -2] \), \( 10 \)
Example 6

The depth, x metres, at a particular point of a river at time t hours after midnight for a 24-hour period is given by 
\[ x(t) = 4 \sin \frac{\pi t}{6} + 5, \quad 0 \leq t \leq 24. \]

a Write down the minimum and maximum depths of the river.
b What is the period of x in hours?
c Find the exact depth at 4 pm.
d Sketch the graph of x against t.
e A ship needs a depth of at least 7 metres to travel down the river. Find the times of the day when the ship can travel past this point of the river.

Solution

a The amplitude is 4 and the mean position is 5. So the range of the function is [1, 9] and, hence, the minimum depth is 1 metre and maximum depth is 9 metres.
b \[ P = \frac{2\pi}{\pi/6} = 2\pi \times \frac{6}{\pi} = 12, \] so the period is 12 hours.
c At 4 pm, \( t = 16. \)
\[ x(16) = 4 \sin \frac{16\pi}{6} + 5 \]
\[ = 4 \sin \frac{8\pi}{3} + 5 \]
\[ = 4 \sin \frac{2\pi}{3} + 5 \]
as \( \frac{8\pi}{3} \) and \( \frac{2\pi}{3} \) are both in the second quadrant, 2\( \pi \) apart.
\[ x(16) = 4 \times \frac{\sqrt{3}}{2} + 5 \]
\[ = 2\sqrt{3} + 5 \]
The exact depth at 4 pm is \((2\sqrt{3} + 5)\) metres.
d When sketching the graph, we must label the axes x and t and only sketch the graph for the given domain of \( 0 \leq t \leq 24. \) Note that as this function has a restricted domain, indicate the endpoints as shown below.

e We need to solve \( 4 \sin \frac{\pi t}{6} + 5 = 7. \) This simplifies to \( \sin \frac{\pi t}{6} = \frac{1}{2}. \)

Now \( \sin \frac{\pi}{6} = \frac{1}{2} \) and \( \sin \theta > 0 \) in the first and second quadrants.
\[ \frac{\pi t}{6} = \frac{\pi}{6}, \frac{\pi}{6} - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6} \]
\[ = \frac{\pi}{6}, \frac{\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \]
\[ t = 1, 5, 13, 17 \]

Using these values and the graph, we see that the ship can travel past this point of the river between the hours of 1 am and 5 am and between 1 pm and 5 pm.

A graphics calculator can be used to check the shape of the graph in example 6 part d and to solve the equation in part e. For example, the calculator screen to the right shows the first intersection point of the graph above with the line with equation \( y = 7. \)
exercise 2.3

8 The height above the ground, $x$ metres, of a person on a ride at a carnival at time $t$ seconds after the ride started moving can be modelled by the function

$$x(t) = 6 \cos \frac{\pi t}{24} + 6, \quad t \geq 0.$$  

a. How long, in seconds, does the ride take to complete a revolution?  
b. What is the exact height of the ride when $t = 20$?  
c. Write down the minimum and maximum heights reached in metres.  
d. Sketch the graph of $x$ versus $t$, $0 \leq t \leq 120$.  
e. For what values of $t$, during the first 60 seconds of the ride, is the height at most 3 metres?

9 The surface of a speed hump is modelled by the function $y(x) = 12 \sin \frac{\pi x}{16}, 0 \leq x \leq 16$ where $y$ cm is the height above the road surface at a distance $x$ cm from the left-most edge of the speed hump.

a. Find the exact height of the speed hump 12 cm from the left-most edge of the speed hump.  
b. For what values of $x$ is the height of the speed hump more than 6 cm?  
c. Sketch the graph of $y$ versus $x$.

10 The temperature $T$°C at a town in central Australia at time $t$ hours after 8 am can be modelled by $T(t) = 15 \sin \frac{\pi t}{12} + 20, \quad t \geq 0$.  

a. What is the minimum temperature in °C, and at what time of day does it occur?  
b. What is the maximum temperature in °C, and at what time of day does it occur?  
c. Find the temperature in °C at 6 pm.  
d. Sketch the graph of $T$ against $t$.  
e. During what times of the day is the temperature below 15°, correct to the nearest minute?

11 The height above the ground, $x$ metres, of a particle that is suspended by elastic from the ceiling, at time $t$ seconds after the particle is released, can be modelled by the function

$$x(t) = 1.2 \cos \frac{t}{3} + 3.5, \quad t \geq 0.$$  

a. How high, in metres, is the particle when it is released?  
b. What is the minimum height, in metres, of the particle?  
c. Exactly how many seconds elapse between consecutive times that the particle is at the minimum height?  
d. Sketch the graph of $x$ against $t$ for $0 \leq t \leq 10$.  
e. For what exact value of $t$ does the height first equal 2.9 metres?
Tangent graphs

Recall that \( \tan x = \frac{\sin x}{\cos x} \), so the tangent function with the rule \( y = \tan x \) does not exist when \( \cos x = 0 \). There are asymptotes wherever \( \cos x = 0 \), i.e. at \( x = \ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \), which can be written as \( x = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z} \). The domain of the tangent function is \( \mathbb{R} \setminus \left\{ \frac{(2k+1)\pi}{2}, k \in \mathbb{Z} \right\} \).

Part of the graph of \( y = \tan x \), for \( -2\pi \leq x \leq 2\pi \), is shown.

The range of the function with rule \( y = \tan x \) is \( \mathbb{R} \). The concept of an amplitude does not exist for a tangent graph.

The period of the function with rule \( y = \tan x \) is \( \pi \). The graph repeats with an interval of \( \pi \).

For a function with equation of the form \( y = \tan nx \), the period \( P = \frac{\pi}{n} \).

Example 7

Sketch the graph of \( y = \tan 2x \) for \( 0 \leq x \leq 2\pi \).

Solution

\( P = \frac{\pi}{n} = \frac{\pi}{2} \).

As \( \tan 2x = \frac{\sin 2x}{\cos 2x} \), there are asymptotes when \( \cos 2x = 0 \), i.e. at \( x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots \).

exercise 2.3

12 Sketch the graphs of:
   - a \( y = \tan 3x \) for \( 0 \leq x \leq 2\pi \)
   - b \( y = \tan \frac{x}{2} \) for \( -2\pi \leq x \leq 2\pi \)
   - c \( y = \tan \frac{x}{4} \) for \( -2\pi < x < 2\pi \)
   - d \( y = \tan 4x \) for \( -\pi \leq x \leq \pi \)

13 Sketch the graphs of each of the following for \( x \geq 0 \), showing 2 complete cycles.
   - a \( y = -\tan x \)
   - b \( y = \tan \frac{\pi x}{2} \)
   - c \( y = 2\tan 3x \)
   - d \( y = -3\tan 2\pi x \)

14 Let \( y = \tan ax \), where \( a > 0 \).
   - a If the period is 2, find the value of \( a \).
   - b If the graph has an asymptote at \( x = 2 \), find all possible values of \( a \).
Tom is marooned on an island in a region that has extreme tides. He is able to accurately measure the depth of the water around the island and finds that it can be modelled by the function \( x(t) = 2.2 \cos \left( \frac{\pi t}{12} \right) + 2.6 \), where \( x \) metres is the depth of the water at time \( t \) hours after 12 noon.

**Part 1—a few simple calculations**

a. What is the period of the function in hours?

b. What is the minimum depth of the water in metres?

c. What is the maximum depth of the water in metres?

**Part 2—a chance to be rescued**

A reef joins the island Tom is on to a neighbouring island. At any time, the depth of water over the reef is the same at all points on the straight path between the islands. This depth is still given by the model. Tom reasons that he will have a greater chance of being rescued if he can make it to the other island. Being unable to swim, he needs the water to be no deeper than 1.5 metres in order to complete the crossing.

d. Sketch the first period of the graph of \( x \) against \( t \), indicating on your graph the times available to Tom to complete the crossing.

e. How much time in hours does Tom have to complete the crossing?

**Part 3—if at first you don’t succeed …**

To his horror, Tom estimates that it will take him more than 10 hours to complete the crossing. He decides that his best chance of being rescued is to build a fire at the high point of the island to attract a passing ship. Tom calculates that the type of ship that passes by his island needs a depth of at least \( 1.1 \sqrt{3} + 2.6 \) (\( \approx 4.5 \)) metres so that it does not run aground.

f. Between what times of day can a ship approach the island?

g. If the ship must travel 10 km to reach the island, what speed in km/hour must the ship average to save Tom?
Absolute value function

The definition of $|x|$
The absolute value of $x$ is written as $|x|$. It is defined by:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$|x|$ can be thought of as the distance that $x$ is from zero. For example, the distance that 3 is from zero is 3, whereas the distance that $-2$ is from zero is 2. So $|3| = 3$, $|-2| = 2$.

$|x|$ is sometimes referred to as the magnitude of $x$, or the modulus of $x$, which can be loosely taken to mean the size of $x$, ignoring any sign or units.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$.

Then $f$ is called the absolute value function. It is an example of a hybrid function; it has different rules for different parts of its domain.

The graph of $y = |x|$
The graph of the absolute value function $f$ is equivalent to the graphs of $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$ sketched on the same set of axes. The graph of $y = |x|$ is shown.

We see from the graph that the range of $f$ is $[0, \infty)$.

Inequalities
Suppose that for the function with the rule $y = |x|$ it is required to find the values of $x$ for which $y < 3$ i.e. $|x| < 3$.

From the graph, $y = 3$ or $|x| = 3$ when $x = -3$ or $x = 3$. So $y < 3$ for $-3 < x < 3$ or, equivalently, for $x \in (-3, 3)$.

Similarly, the values of $x$ for which $y > 3$, i.e. $|x| > 3$, are $x < -3$ or $x > 3$. This can be written as $x \in \mathbb{R} \setminus [-3, 3)$, where the square brackets are required to ensure that the numbers from $-3$ to 3 inclusive are removed from $\mathbb{R}$.

An absolute value equation can be transformed and solved as follows:

$|x| = 3$

$x^2 = 9$ (squaring both sides)

$x = \pm 3$

To solve an inequality such as $|x| < 3$, first solve the equality and then use a graph as an aid, or solve it algebraically by first squaring and then using the method in section 2.1.
### Example 1

**a** Sketch the graph of \( y = |x + 2| \).

**Solution**

**a** We use the definition \( |a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases} \)

In this case we have \( a = x + 2 \), so \( y = |x + 2| = \begin{cases} x + 2 & x + 2 \geq 0 \\ -(x + 2) & x + 2 < 0 \end{cases} \)

This is equivalent to \( y = |x + 2| = \begin{cases} x + 2 & x \geq -2 \\ -(x + 2) & x < -2 \end{cases} \)

Sketch each of the straight line graphs for the appropriate values of \( x \).

**b** First solve the equation \( |x + 2| = 5 \).

\[
(x + 2)^2 = 25
\]

\[
x + 2 = \pm 5
\]

\[
x = -7, 3
\]

As we require \( \{x: |x + 2| > 5\} \), the graph shows that \( x < -7 \) or \( x > 3 \).

So the solution is \((-\infty, -7) \cup (3, \infty)\).

### Example 2

Sketch the graph of \( y = |x| - 4 \) and state the \( x \)-intercepts. Give the range of the function.

**Solution**

In this case \( y = |x| - 4 = \begin{cases} x - 4 & x \geq 0 \\ -x - 4 & x < 0 \end{cases} \)

Sketch each of the straight line graphs for the appropriate values of \( x \).

The \( x \)-intercepts can be read from the graph as \((-4, 0)\) and \((4, 0)\).

Alternatively they can be found algebraically as follows:

\[
|x| - 4 = 0
\]

\[
|x| = 4
\]

\[
x = \pm 4 \quad \text{(as \( -4 = 4 \) and \( 4 = 4 \))}
\]

The range of the function is \([-4, \infty)\).

**tip**

Note that there is a very strong analogy between the translations of absolute value graphs and parabolas. The turning point of the graph of \( y = (x + b)^2 + c \) is \((-b, c)\); the vertex of the graph of \( y = |x + b| + c \) is \((-b, c)\).
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Example 3
Sketch the graph of \( y = -|x - 2| + 1 \).

Solution
First note that the graph of \( y = -|x| \) will be a reflection in the \( x \)-axis of the graph of \( y = |x| \). Then, from the previous tip, the vertex of the graph is at \((2, 1)\).

exercise 2.4
1. a Sketch the graph of \( y = |x + 3| \).
   b Use algebra to find \( \{x : |x + 3| < 7\} \).

2. a Sketch the graph of \( y = |x - 1| \).
   b Use algebra to find \( \{x : |x - 1| > 3\} \).

3. a Sketch the graph of \( y = |x - 4| \).
   b Use algebra to find \( \{x : |x - 4| \geq 6\} \).
   c Find the \( x \)-intercepts.

4. a Sketch the graph of \( y = |x| - 5 \).
   b Find the \( x \)-intercepts.
   c State the range of the function.

5. a Sketch the graph of \( y = |x| - 3 \).
   b Find the \( x \)-intercepts.
   c State the range of the function.

6. a Sketch the graph of \( y = |x| + 2 \).
   b State the range of the function.

7. Sketch the graphs of:
   a \( y = -|x + 1| - 2 \)
   b \( y = -|x - 1| - 3 \)
   c \( y = -|x + 2| + 1 \)

Absolute values with other functions
Absolute values used in conjunction with other functions give rise to some interesting graphs.

Example 4
Sketch the graph of \( y = |x^2 + x - 2| \) by first finding the \( x \)-intercepts.

Solution
The \( x \)-intercepts are given by \( |x^2 + x - 2| = 0 \), so:
\[
x^2 + x - 2 = 0
\]
\[
(x + 2)(x - 1) = 0
\]
\[
x = -2, 1
\]
The \( x \)-intercepts are \((-2, 0)\) and \((1, 0)\).
To sketch the graph, there are two main approaches.
Method 1
Using the definition, \(|x^2 + x - 2| = \begin{cases} x^2 + x - 2 & x^2 + x - 2 \geq 0 \\ -(x^2 + x - 2) & x^2 + x - 2 < 0 \end{cases}\)
This is equivalent to \(|x^2 + x - 2| = \begin{cases} x^2 + x - 2 & x \in (-\infty, -2] \cup [1, \infty) \\ -(x^2 + x - 2) & x \in (-2, 1) \end{cases}\)
Each of the corresponding parabolas can be sketched for the appropriate values of \(x\).

Method 2
The graph of \(y = x^2 + x - 2 = (x + 2)(x - 1)\) is a parabola with \(x\)-intercepts at \((-2, 0)\) and \((1, 0)\). The absolute value of a positive number is equal to that number, and the absolute value of a negative number is equal to the negative of that number (and is therefore positive). Hence to sketch the graph of \(y = |x^2 + x - 2|\), simply sketch the graph of \(y = x^2 + x - 2\), and reflect in the \(x\)-axis the part of the graph that has negative \(y\) values. The graph is shown below.

**Example 5**

Sketch the graph of \(y = |(2 - x)(x - 1)(x + 3)|\) by first finding the \(x\)-intercepts.

**Solution**
The \(x\)-intercepts are \((-3, 0)\), \((1, 0)\) and \((2, 0)\). The graph of \(y = (2 - x)(x - 1)(x + 3)\) is a cubic graph for which the \(y\) values are negative for \(x \in (-3, 1) \cup (2, \infty)\). Reflect those parts of the graph in the \(x\)-axis to get the graph shown below.
Example 6

Sketch the graph of \( y = \frac{1}{|x - 1|} \) by first finding the asymptotes.

Solution

The asymptotes are \( x = 1 \) and \( y = 0 \). The graph of \( y = \frac{1}{x - 1} \) is a hyperbola whose \( y \) values are negative for \( x < 1 \). Reflect this part of the graph in the \( x \)-axis to get the graph as shown.

Example 7

Sketch the graph of \( y = \log_e |x + 2| \) by first finding the asymptote.

Solution

The asymptote is \( x = -2 \). The domain of \( y = \log_e (x + 2) \) is \((-2, \infty)\), as we cannot take the log of zero or a negative number. However, \(|x + 2| \geq 0\) so the domain of \( y = \log_e |x + 2| \) is \( R \setminus \{-2\} \). We know that \(|a| = |-a|\) so \( \log_e |a| = \log_e |-a| \). The graph is symmetrical about the asymptote as shown.

Example 8

Sketch the graph of \( y = |\log_e (x + 1)| \) by first finding the asymptote.

Solution

The asymptote is \( x = -1 \). The domain of \( y = \log_e (x + 1) \) is \((-1, \infty)\). The graph of \( y = \log_e (x + 1) \) is a log graph whose \( y \) values are negative for \( x < 0 \). Reflect this part of the graph in the \( x \)-axis to get the graph as shown.
Example 9

Sketch the graph of $y = |\sin x|$ for $0 \leq x \leq 2\pi$.

Solution

The $x$-intercepts are $(0, 0), (\pi, 0)$ and $(2\pi, 0)$. The $y$ values of $y = \sin x$, $x \in [0, 2\pi]$ are negative for $x \in (\pi, 2\pi)$. Reflect that part of the graph in the $x$-axis to get the graph shown here.

exercise 2.4

8 Sketch the following graphs by first finding the $x$-intercepts.
   a $y = |x^2 - 2x - 8|
   b y = |x^2 + 3x - 10|

9 Sketch the following graphs by first finding the $x$-intercepts.
   a $y = |(x - 1)(x + 1)(x + 3)|
   b y = |(x + 1)(x + 3)(x + 4)|

10 Sketch the following graphs by first finding the $x$-intercepts.
   a $y = |(x - 1)^2(x - 3)|
   b y = |(x + 2)^2(x - 2)|

11 Sketch the following graphs by first finding the asymptotes.
   a $y = \frac{1}{|x - 3|}
   b y = \frac{1}{|x + 2|}$

12 Sketch the following graphs by first finding the asymptotes.
   a $y = \log_e|x + 1|
   b y = \log_e|x - 3|$

13 Sketch the following graphs by first finding the asymptotes.
   a $y = |\log_e(x - 3)|
   b y = |\log_e(x + 2)|$

14 Sketch the graphs of:
   a $y = |\cos x|, x \in [0, 2\pi]
   b y = |\sin 2x|, x \in [0, 2\pi]
   c $y = |\cos 2x|, x \in [0, 2\pi]
   d $y = |\sin \frac{x}{2}|, x \in [-\pi, \pi]$

15 Sketch the graphs of:
   a $y = |\tan x|, x \in [0, 2\pi]
   b y = |\tan 2x|, x \in [0, 2\pi]
   c $y = |\tan \frac{x}{2}|, x \in [-2\pi, 2\pi]$
Summary

Power functions and their graphs

A power function has the general rule $y = x^n$. Here we considered cases where $n \in N$ or $n = \frac{1}{2}, -1, -2$.

For power functions when $n \in N$, the function is that of the simplest polynomial of degree $n$ (e.g. if $n = 3$, the function is $y = x^3$, which is the simplest polynomial of degree 3).

If $n = \frac{1}{2}, y = x^{1/2}$ and the graph of the square root function, $y = \sqrt{x}$, is:

If $n = -1, y = x^{-1}$ or $y = \frac{1}{x}$, and the graph of the power function $y = x^{-1} = \frac{1}{x}$ is a hyperbola:

If $n = -2, y = x^{-2}$ or $y = \frac{1}{x^2}$, and the graph of the power function $y = x^{-2} = \frac{1}{x^2}$ is a truncus:

Polynomial functions in factorised form

A polynomial function that is written as the product of linear factors will have a graph with $x$-intercepts at the $x$ values for which each of the linear factors is zero.

Exponential and logarithmic functions

An exponential function has rule $y = a^x, a \in R^* \setminus \{1\}$.
Functions and their graphs

An exponential function with rule \( y = a^x \) has the following properties:

- Its graph passes through \((0, 1)\) as \( a^0 = 1 \).
- Its graph passes through \((1, a)\) as \( a^1 = a \).
- Its graph has a horizontal asymptote at \( y = 0 \).
- The domain of the function is \( \mathbb{R} \).
- The range of the function is \( \mathbb{R}^+ \).

A logarithmic function has rule \( y = \log_a x, a \in \mathbb{R}^+ \setminus \{1\} \).

A logarithmic function with rule \( y = \log_a x \) has the following properties:

- Its graph passes through \((1, 0)\) as \( \log_a 1 = 0 \) (as \( a^0 = 1 \)).
- Its graph passes through \((a, 1)\) as \( \log_a a = 1 \) (as \( a^1 = a \)).
- Its graph has a vertical asymptote at \( x = 0 \) (\( \log_a x \) is only defined for \( x > 0 \)).
- The domain of the function is \( \mathbb{R}^+ \).
- The range of the function is \( \mathbb{R} \).
- The graphs of \( y = \log_2 x \) and \( y = \log_{10} x \) are shown below.

Circular functions and their graphs

- When sketching graphs, the basic shapes are:
  \( y = \sin x \) and \( y = \cos x \)

- \( y = a \sin nx + c \) and \( y = a \cos nx + c \) both have period \( \frac{2\pi}{n} \). If \( a \) is positive, the amplitude is \( a \); if \( a \) is negative, the amplitude is \( -a \).
- \( y = a \tan nx + c \) has period \( \frac{\pi}{n} \). Two of the asymptotes have equations \( x = \pm \frac{\pi}{2n} \). Others can be found by adding integer multiples of the period.
Absolute value function

The absolute value of \( x \) is written as \(|x|\) and defined as:

\[ |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \]

The graph of the absolute value function is equivalent to the graphs of \( y = x, x \geq 0 \), and \( y = -x, x < 0 \), sketched on the same set of axes. The graph of \( y = |x| \) is shown here.

Revision questions

Short answer

1. Sketch the graph of:
   - \( a \) \( y = (x - 3)^2 \)
   - \( b \) \( y = (x - 3)(x + 4) \)
   - \( c \) \( y = x^2 + 2x - 15 \)

2. The parabola below has an equation of the form \( y = ax^2 + bx + c \). Use the \( x \)- and \( y \)-intercepts to find the equation of the parabola.

3. Sketch the graph of:
   - \( a \) \( y = x(x - 3)(x + 2) \)
   - \( b \) \( y = (x - 1)^2(x + 6) \)

4. Sketch the graph of:
   - \( a \) \( y = (x - 1)(x + 1)(x + 2)(x - 3) \)
   - \( b \) \( y = (x + 3)(x + 1)(x - 2)^2 \)
   - \( c \) \( y = (x + 1)^2(x - 2)^2 \)
   - \( d \) \( y = (x + 1)^3(x - 2) \)

5. \( a \) The graph of a cubic function has a turning point at \((a, 0)\) and a second \( x \)-intercept at \((b, 0)\). If its \( y \)-intercept is \((0, -a^2b)\), find its equation in factorised form.
   - \( b \) The graph of a polynomial has a stationary point of inflection at \((-a, 0)\) and a turning point at \((b, 0)\). Its \( y \)-intercept is \((0, 2a^2b^2)\). Find its equation in factorised form, given that it has the lowest possible degree.

6. Sketch the graph of:
   - \( a \) \( y = 4^x \)
   - \( b \) \( y = 4 \times 2^x \)
   - \( c \) \( y = 6^{-x} \)
   - \( d \) \( y = e^{2x} \)

7. Sketch the curve with equation below, labelling the \( y \)-intercept with its exact value in each case.
   - \( a \) \( y = e^{x^3} \)
   - \( b \) \( y = e^{x^4} \)
8 Sketch the curve with equation below, labelling the \(x\)-intercept with its exact value in each case.

a \(y = \log_e 6x\)  
b \(y = \log_e (x - 2)\)  
c \(y = \log_e (2x - 3)\)

9 Sketch the graph of each of the following for \(0 \leq x \leq 2\pi\).

a \(y = 4 \sin x\)  
b \(y = 3 \cos x - 4\)  
c \(y = 2 \sin 3x\)  
d \(y = \tan \frac{x}{2}\)

10 Given that \(a, b, c > 0\), find a relationship between some or all of \(a, b, c\) such that the graph of \(y = a \cos bx + c\) meets the \(x\)-axis only once per period.

11 The graph of \(y = 2\sin bx, 0 \leq x \leq \frac{\pi}{2}\), intersects the line with equation \(y = \sqrt{3}\) in two points. The rightmost intersection has coordinates \(\left(\frac{\pi}{4}, \sqrt{3}\right)\).

a Find the coordinates of the leftmost intersection.  
b Find the exact value of \(b\).

12 a Sketch the graph of \(y = |x + 6|\).  
b Use algebra to find \(\{x : |x + 6| \geq 8\}\).

13 a Sketch the graph of \(y = |x| - 7\).  
b Find the \(x\)-intercepts.  
c State the range of the function.

14 Sketch the following graphs by first finding the \(x\)-intercepts.

a \(y = |(x - 3)(x + 2)|\)  
b \(y = |x^2 - 3x - 4|\)

15 Sketch the graphs of the following by first finding the \(x\)-intercepts.

a \(y = |(x - 1)(x + 2)(x + 4)|\)  
b \(y = |(x + 2)^2(x - 1)|\)

16 Sketch the graphs of the following by first finding the asymptotes.

a \(y = \frac{1}{|x - 4|}\)  
b \(y = \frac{1}{|x + 3|}\)

17 Sketch the graphs of the following by first finding the asymptote.

a \(y = \log_e |x - 5|\)  
b \(y = |\log_e (x + 3)|\)

18 Sketch the graph of:

a \(y = |\cos 3x|, 0 \leq x \leq 2\pi\)  
b \(y = |\tan \frac{x}{4}|, -2\pi \leq x \leq 2\pi\)

**Extended response**

The graph at right has an equation of the form \(y = ax^3 + bx^2 + cx + d\).

a Use the \(x\)-intercepts to find its equation in factorised form. (Hint: \(a\) is a common factor.)

b Use the \(y\)-intercept to find the value of \(a\).

c Find the equation of the cubic graph in the form \(y = ax^3 + bx^2 + cx + d\).

d Find the \(x\) values for which \(y = 10\), correct to 2 decimal places.

e Find the coordinates of the maximum turning point, correct to 2 decimal places.
2. The intensity of light, \( I \) units, at a depth of \( x \) metres beneath the surface of a liquid can be modelled by \( I(x) = I_0 e^{-kx} \), \( x \geq 0 \). The intensity is 20 units at the surface and is halved at a depth of 3 metres.

   a. State the value of \( I_0 \).
   b. Find the exact value of \( k \).
   c. Sketch the graph of \( I \) versus \( x \).
   d. To the nearest centimetre, at what depth will the intensity be 7 units?

3. The height from the ground, \( x \) metres, of a swing at time \( t \) seconds after the swing was released, can be modelled by the function \( x(t) = 0.5 \cos \frac{\pi}{t} + 0.8 \), \( 0 \leq t \leq 5 \).

   a. How long in seconds does the swing take to reach its original height for the first time after its release?
   b. What is the exact height of the swing in metres when \( t = 0.25 \)?
   c. Write down the minimum and maximum heights reached in metres.
   d. Sketch the graph of \( x \) versus \( t \).
   e. For what values of \( t \), between the time of release and the next time it reaches its original height, is the height at least 1.05 metres?